CME LIVE OPEN ACCESS
Journal of
Theoretical Physics & Mathematics Research
Research Article
ISSN: 3065-8802

A Note on Some Classes of Function Algebras

M E Egwe*

Department of Mathematics, University of Ibadan, Nigeria.

***Corresponding Author:** M E Egwe, Department of Mathematics, University of Ibadan, Nigeria.

Received: 2025 Nov 12 **Accepted:** 2025 Dec 01 **Published:** 2025 Dec 12

Abstract

In this note, an attempt is made at highlighting a construct of the algebras of holomorphic functions on the unit disc, bi-invariant functions, Lie pseudo groups and the Colombeau's algebra (of generalized functions). We establish some new results on these algebras and also give new proofs to some existing ones.

Keywords: Operator Algebras, Holomorphic Functions, Lie Pseudo Groups, Spherical Functions

1. Introduction

The theory of holomorphic functions on the unit disc and its ubiquity abound in literature [1-21]. Here, we shall concern ourselves with introducing some of the basic definitions needed in the succeeding developments of some function algebras and proving several essential algebraic results.

Definition 1.1. A linear space A over C is said to be an algebra if it is equipped with a binary operation, referred to as multiplication and denoted by juxtaposition, from A × A → A such that

$$f(gh) = (fg)h, \tag{1.1}$$
$$f(g+h) = fg + fh;\ (g+h)f = gf + hf, \tag{1.2}$$
$$a(fg) = (af)g = f(ag). \tag{1.3}$$

for all $f, g, h \in \mathcal{A}$ and $a \in \mathbb{C}$.

$\mathcal{A}$ is called a commutative algebra if $\mathcal{A}$ is an algebra and

$$fg = gf \tag{1.4}$$

$$ef = fe = f \tag{1.5}$$

for all $f \in \mathcal{A}$.

Definition 1.2. A normed linear space$(\mathcal{A}, \|\cdot\|)$ over $\mathbb{C}$ is said to be normed algebra if $\mathcal{A}$ is an algebra and

$$\|fg\| \leq \|f\|\,\|g\|$$

for all $f, g \in \mathcal{A}$.

A normed algebra $\mathcal{A}$ is said to be a Banach algebra if the normed linear space $(\mathcal{A}, \|\cdot\|)$ is a Banach space.

Definition 1.3. Let $\mathcal{A}$ be a complex algebra. An involution on $\mathcal{A}$ is a mapping $* : f \to f^*$ from $\mathcal{A}$ into $\mathcal{A}$ satisfying the following conditions.

$$(f+g)^* = f^* + g^*, \tag{1.6}$$
$$(\lambda f)^* = \overline{\lambda} f^*, \tag{1.7}$$
$$(fg)^* = g^* f^*, \tag{1.8}$$
$$(f^*) = f, \tag{1.9}$$

for all $f, g \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. $\mathcal{A}$ is then called a $*-$algebra or an algebra with involution.

A C^*-algebra is a Banach algebra $\mathcal{A}$ with involution in which for all $f \in \mathcal{A}$,

$$\|f^* f\| = \|f\|^2 . \tag{1.10}$$

A homomorphism between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a linear map $\varphi : \mathcal{A} \to \mathcal{B}$ that satisfies $\varphi(fg) = \varphi(f)\varphi(f)$ and $\varphi(f^*) = \varphi(g)^*$ for all $f \in \mathcal{A}$ and $g \in \mathcal{B}$ [13]. An isomorphism between two C^*-algebras is an invertible homomorphism.

The dual $\mathcal{A}'$ of a $*-$algebra $\mathcal{A}$ has a canonical involution given by $f^*(x) = \overline{f(x^{-1})}$ for all $f \in \mathcal{A}$.

]**Definition 1.4.** A function algebra on a compact Hausdorff space X is a commutative Banach algebras $\mathcal{A}$ over $\mathbb{C}$ which satisfies the following conditions:

1. The elements of $\mathcal{A}$ are continuous complex-valued functions defined on X, i.e. $\mathcal{A} \subset C(X)$;
2. $\mathcal{A}$ contains all constant functions on X;
3. The operations on $\mathcal{A}$ are the pointwise additions and multiplication;
4. $\mathcal{A}$ is closed with respect to the uniform norm in $C(X)$
$$\|f\| = \sup_{x\in X}|f(x)|, \ f \in \mathcal{A};$$
5. $\mathcal{A}$ separates the points of X.

Function algebras are commutative Banach algebras over $\mathbb{C}$ with unit (the constant function 1 on X) and this fact plays a crucial role in their studies [11].

Let X be a compact Hausdorff space and $\mathcal{A}$ a non-empty subset of $C(X)$. For each $x \in X$ the evaluation map at x, denoted by ϕ_x, is defined by

$$\phi_x(f) = f(x) \text{ for} f \in \mathcal{A} \tag{1.11}$$

We observe that if $\mathcal{A}$ is a subspace, then $\phi_x : \mathcal{A} \to \mathbb{C}$ is a linear map, and if $\mathcal{A}$ is subalgebra, then ϕ_x is a homomorphism. If $\mathcal{A}$ contains 1 then $\phi_x(1) = 1$ and hence $\phi_x \neq 0$.

Unitization of normed algebras is well-known [9].

Definition 1.5. Let $\mathcal{A}$ be a normed algebra. A left (right) approximate identity for $\mathcal{A}$ is a net $(e_\alpha)_\alpha$ in $\mathcal{A}$ such that $e_\alpha f \to f$ $(fe_\alpha \to f)$ for each $f \in \mathcal{A}$. An approximate identity for $\mathcal{A}$ is a net (e_α) which is both a left and a right approximate identity. A (left or right) approximate identity $(e_\alpha)_\alpha$ is bounded by $M > 0$ if $\|e_\alpha\| \leq M$ for all α. A has left (right) approximate units if, for each $f \in \mathcal{A}$ and $\epsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|f - uf\| \leq \epsilon$ $(\|f - fu\| \leq \epsilon)$, and $\mathcal{A}$ has an approximate unit if, for each $f \in \mathcal{A}$ and $\epsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|f - uf\| \leq \epsilon$ and $\|f - fu\| \leq \epsilon$. A has a (left, right) approximate unit bounded by $M > 0$, if the elements u can be chosen such that $\|u\| \leq M$.

Suppose that $\mathcal{A}$ is an approximately unital Banach algebra. We define the unitization of $\mathcal{A}$ by considering the canonical 'left regular representation' $\lambda : \mathcal{A} \to B(\mathcal{A})$, and identifying $\mathcal{A} + \mathbb{C}_1$ with the span of $\lambda(\mathcal{A}) + \mathbb{C}I_{\mathcal{A}}$, which is a unital Banach subalgebra of $B(\mathcal{A})$. Thus if $f \in \mathcal{A}$ and $\mu \in \mathbb{C}$ then

$$\|f + \mu 1\| = \sup_{\|c\|\leq 1} \|fc + \mu c\| , \tag{1.12}$$

for all $c \in \mathcal{A}$.

It is occasionally useful that there are some other equivalent expressions for the quantity above. For example, if $(e_\alpha)_\alpha$ is an approximate identity for $\mathcal{A}$ then

$$\|f+\mu_1\| = \lim_{\alpha} \|fe_\alpha + \mu e_\alpha\| = \sup_{\alpha} \|fe_\alpha + \mu e_\alpha\| . \tag{1.13}$$

Given a real algebra $\mathcal{A}$, the complexification $\mathcal{B}$ of $\mathcal{A}$ is the set $\mathcal{A}\times\mathcal{A}$ with the operations of addition, multiplication, and scalar multiplication defined by

$$\begin{aligned}(f_1,g_1)+(f_2,g_2) &:= (f_1+f_2, g_1+g_2)\,,\\ (\alpha+i\beta)(f,g) &:= (\alpha f-\beta g, \alpha g+\beta f)\,,\\ (f_1,g_1)(f_2,g_2) &:= (f_1f_2-g_1g_2, f_1g_2+g_1f_2)\,,\end{aligned}$$

for all $f_1, f_2, g_1, g_2 \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$.

Definition 1.6. An operator $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ between two Hilbert space is simply a linear map (i.e., $\varphi(\lambda f + \mu g) = \lambda\varphi(f) + \mu\varphi(g)$ for all $\lambda, \mu \in \mathbb{C}$ and $f, g \in \mathcal{H}_1$.

An operator on a Hilbert space $\mathcal{H}$ is bounded if and only if it is continuous in the sense that $f_n \to f$ implies $\varphi f_n \to \varphi f$ for all convergent sequences (f_n) in $\mathcal{H}$.

Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be an operator. We define $\|\varphi\| \in \mathbb{R}^+ \cup \{\infty\}$ by

$$\|\varphi\| := \sup_{\|f\|_{\mathcal{H}_1}=1} \|\varphi(f)\|_{\mathcal{H}_2}\,, \tag{1.14}$$

for all $f \in \mathcal{H}_1$, where $\|f\|_{\mathcal{H}_1} = |\langle f, f\rangle|_{\mathcal{H}_1}$, etc. We say that φ is bounded when $\|\varphi\| < \infty$, in which case the number $\|f\|$ is called the norm of φ.

1.1 Some Classes of Operator Algebras

(1) Operator Algebra Valued Continuous Functions: Let X be a compact space, and $\mathcal{A}$ an operator algebra. Then the operator space $\mathcal{C}(X;\mathcal{A})$ is an operator algebra when equipped with the product defined by $(fg)(t) = f(t)g(t)$. For example, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ then $\mathcal{C}(X;\mathcal{A})$ is a subalgebra of the $\mathcal{C}^*-$algebra $\mathcal{C}(X;\mathcal{B}(\mathcal{H}))$. If $\mathcal{A}$ is unital, then $\mathcal{C}(X;\mathcal{A})$ is also unital, the identity being the constant function equal to the identity of $\mathcal{A}$.

(2) Uniform Algebras: A (concrete) uniform algebra is a unital-subalgebra of $\mathcal{C}(X)$, for some compact space X. Here, we will consider any uniform algebra as endowed with its minimal operator space structure. Then an (abstract) uniform algebra is a unital operator algebra which is completely isometrically isomorphic to a concrete uniform algebra. In this way, we regard uniform algebras as a subclass of the operator algebras.

More generally, we will use the term *function algebra* for an operator algebra $\mathcal{A}$ for which there exists a compact space X and a completely isometric homomorphism $\pi : \mathcal{A} \to \mathcal{C}(X)$. Any function algebra is a minimal operator space.

(3) Disc Algebras: This fundamental example of a uniform algebra has two equivalent definitions. Let us denote by $\mathbb{B}$ and $\mathbb{T}$ the open unit disc of $\mathbb{C}$ and the unitary complex group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ respectively. Then the disc algebra $\mathcal{A}(\mathbb{B})$ is the subalgebra of $\mathcal{C}(\overline{\mathbb{B}})$ consisting of all continuous functions $F : \overline{\mathbb{B}} \to \mathbb{C}$, whose restriction to $\mathbb{B}$ is holomorphic. By the maximal modulus theorem, the restriction of functions in $\mathcal{A}(\mathbb{B})$ to the boundary $\mathbb{T}$ is an isometry. Hence we may alternatively regard $\mathcal{A}(\mathbb{B}) \subset \mathcal{C}(\mathbb{T})$ as a uniform algebra acting on $\mathbb{T}$. In that representation, $\mathcal{A}(\mathbb{B})$ consists of all elements of $\mathcal{C}(\mathbb{T})$ whose harmonic extension to $\mathbb{B}$ given by the Poisson integral is holomorphic. Equivalently, given any $f \in \mathcal{C}(\mathbb{T})$, we associate Fourier coefficients

$$\widehat{f}(k) = \int_{\mathbb{T}} f(z) z^{-k} d\mu(z), \quad k \in \mathbb{Z}. \tag{1.15}$$

Then, $\mathcal{A}(\mathbb{B}) \subset C(\mathbb{T})$ is the closed subalgebra of all $f \in \mathcal{C}(\mathbb{T})$ such that $f(k) = 0$ for every $k < 0$. The (holomorphic) polynomials form a dense subalgebra of $\mathcal{A}(\mathbb{B})$.

(4) Adjoint Algebra: The adjoint operator space $\mathcal{A}^*$ is an operator algebra, with product $a^*b^* = (ba)^*$, for $a, b \in \mathcal{A}$. Indeed, if $\mathcal{A}$ is a subalgebra of a $\mathcal{C}^*-$algebra $\mathcal{B}$, then $\mathcal{A}^*$ may be identified with the subalgebra $a^* : a \in \mathcal{A}$ of $\mathcal{B}$. Note that if $\mathcal{A}$ has a continuous approximate identity (cai) $(e_t)_t$, then $(e_t^*)_t$ is a cai for $\mathcal{A}^*$.

(5) Multiplier Operator Algebras [3]: A left (right) multiplier on $\mathcal{A}$ is a linear mapping $T : \mathcal{A} \to \mathcal{A}$ such that

$$T(xy) = T(x)y \ (= xT(y))$$

for all $x, y \in \mathcal{A}$.

T is called a two-sided multiplier (or simply, a multiplier) on $\mathcal{A}$ if it is a left and a right multiplier.

Let $\mathcal{A}$ be a $\mathcal{C}^*-$algebra. The multiplier algebra of $\mathcal{A}$, denoted by $\mathcal{M}(\mathcal{A})$, is the universal $\mathcal{C}^*-$algebra with the property that $\mathcal{M}(\mathcal{A})$ contains $\mathcal{A}$ as an essential ideal and for any $\mathcal{C}^*-$algebra $\mathcal{B}$ containing $\mathcal{A}$ as an essential ideal there exists a unique $^*-$homomorphism $\pi : \mathcal{B} \to \mathcal{M}(\mathcal{A})$, that is, the identity on $\mathcal{A}$.

If $\mathcal{A}$ is a unital $\mathcal{C}^*-$algebra. Then $\mathcal{M}(\mathcal{A})$ is unital and $\mathcal{M}(\mathcal{A}) = \mathcal{A}$.

1.2 Bi-invariant Function

Let G be a locally compact group and K a compact subgroup. We assume the normalization $\int_K d\mu_k(k) = 1$. The projection

$$P : G \to K \setminus G/K$$

defined by $p(g) = KgK$ identifies $C_c(K \setminus G/K)$ with

$$\{\phi \in C_c(K \setminus G/K) : \phi(k_1 g k_2) = \phi(g) \text{ for } k_i \in K,\ g \in G\}.$$

Now, let us identify $C(K \setminus G/K)$, $\mathcal{C}_\infty(K \setminus G/K)$, and $L^p(K \setminus G/K)$ with the bi-K-invariant functions of the same category on G. In the sequel, $\mathcal{C}_\infty$ means the commutative Banach algebra of continuous functions that vanish at infinity with pointwise multiplication and sup norm. In other words, $\mathcal{C}_\infty(G)$ consists of the continuous functions $f : G \to \mathbb{C}$ such that, if $\epsilon > 0$ then there exists a compact set $C \subset G$ such that $|\phi(x)| < \epsilon$ for all $x \in G \setminus C$.

We now have the projections $C(G) \to C(K \setminus G/K)$, $C_c(G) \to C_c(K \setminus G/K)$, $\mathcal{C}_\infty(G) \to \mathcal{C}_\infty(K \setminus G/K)$ and $L^p(G) \to L^p(K \setminus G/K)$, all denote $\phi \to \phi^\sharp \in C_c(G)^\sharp$ given by

$$\phi^\sharp(g) = \int_K \int_K \phi(k_1 g k_2) d\mu_k(k_1)(k_2). \tag{1.16}$$

Some immediate properties are (see [12]):

- If $\phi_1 \in C(K \setminus G/K)$ and $\phi_2 \in C(G)$ the $(\phi_1\phi_2)^\sharp = \phi_1\phi_2^\sharp$ for the pointwise multiplication, and
- If $\phi_1 \in C_c(K \setminus G/K)$ and $\phi_2 \in C_c(G)$ then $(\phi_2 * \phi_1)^\sharp = \phi_2^\sharp * \phi_1$ and $(\phi_1 * \phi_2)^\sharp = \phi_1^\sharp * \phi_2$ for the convolution product. In particular, $C_c(K \setminus G/K)$ is a subalgebra of the convolution algebra $C_c(G)$ and $L^1(K \setminus G/K)$ is a subalgebra of $L^1(G)$.

We say that (G, K) is Gelfand pair if the convolution algebra $L^1(K \setminus G/K)$ is commutative. If (G, K) is a Gelfand pair, then G/K is a commutative space relative to G, and we also say that (G, K) is a commutative pair. Since $C_c(K \setminus G/K)$ is dense in $L^1(K \setminus G/K)$ it is equivalent to require that $C_c(K \setminus G/K)$ be commutative.

1.3 Holomorphic Functions

As usual, we write $\mathbb{N}$ for the set of natural numbers, $\mathbb{R}$ for the field of real numbers and $\mathbb{C}$ for the field of complex numbers. For any positive integer $n \in \mathbb{N}$ the set $\mathbb{C}^n$ is equipped with the usual vector space structure and for any $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ the norm of z is given by the popular Euclidean norm

$$|z| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}.$$

We define an isomorphism of $\mathbb{R}-$vector spaces between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ by setting $z_j = x_j + iy_j$ for any $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $j = 1, \cdots, n$.

The holomorphic and anti-holomorphic differential operations are given by

$$\begin{cases} \frac{\partial}{\partial_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \frac{1}{i}\frac{\partial}{\partial y_j}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \; j = 1, \cdots, n \\ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial y_j}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \; j = 1, \cdots, n. \end{cases} \tag{1.17}$$

If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$ are multi-indices and $x = (x_1, \cdots, x_n)$ is a point in $\mathbb{R}^n$ then we set

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \; \alpha! = \alpha_1! \cdots \alpha_n!, \; x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \text{ and } D^{\alpha\overline{\beta}} = \frac{\partial^{|\alpha|+|\beta|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \overline{z}_1^{\beta_1} \cdots \partial \overline{z}_n^{\beta_n}}. \tag{1.18}$$

If $\mathbb{B}$ is an open ball in $\mathbb{R}^n$ then we denote by $\mathcal{C}^0(\mathbb{B})$ or $C(\mathbb{B})$ the vector space of complex-valued continuous functions on $\mathbb{B}$ and we denote by $C^k(\mathbb{B})$ the set of k times continuously differentiable functions for any $k \in \mathbb{N}$, $k > 0$. The intersection of the spaces $C^k(\mathbb{B})$ for all $k \in \mathbb{N}$ is the space $C^\infty(\mathbb{B})$ of functions on $\mathbb{B}$ which are differentiable to all orders. It is easy to check that $f \in C^k(\mathbb{B})$ if and only if $D^{\alpha\overline{\beta}} f \in C(\mathbb{B})$ for any pair $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ such that $|\alpha| + |\beta| \leq k$. If $k \in \mathbb{N}$ then the vector space of functions f contained in $C^k(\mathbb{B})$ whose derivatives $D^\alpha f$, $|\alpha| \leq k$, are continuous on D is denoted by $C^k(\mathbb{B})$ and we denote by $C^\infty(\mathbb{B})$ space of infinitely differentiable functions on $\mathbb{B}$ all of whose derivatives are continuous on $\mathbb{B}$ [20].

If $\mathbb{B} \Subset \mathbb{R}^n$ and $f \in C^k\left(\overline{\mathbb{B}}\right)$, $k \in \mathbb{N}$, then we define the C^k norm of f on $\mathbb{B}$ by

$$\|f\|_{k,\mathbb{B}} = \sum_{\alpha \in \mathbb{N}^n |\alpha| \leq k} \sup_{x \in \mathbb{B}} |D^\alpha f(x)|; \tag{1.19}$$

Let $\mathbb{B}$ be an open set in $\mathbb{C}^n$. A complex-valued function f defined on $\mathbb{B}$ is said to be holomorphic on $\mathbb{B}$ if $f \in C^1(\mathbb{B})$ and

$$\frac{\partial f}{\partial \overline{z}_j}(z) = 0 \tag{1.20}$$

for every $z \in \mathbb{B}$ and $j = 1, \cdots, n$. The system of partial differential equations (21) is called the *homogeneous Cauchy–Riemann system*.

The mapping $F : \mathbb{B}_n \to \mathbb{C}^N$, where N is a positive integer, is given by n functions as follows:

$$F(z) = (f_1(z), \cdots, f_n(z)), \; z \in \mathbb{B}.$$

We say that F is a holomorphic mapping if each f_k is holomorphic in $\mathbb{B}$.

It is clear that any holomorphic mapping $F : \mathbb{B} \to \mathbb{C}^n$ has a Taylor type expansion

$$F(z) = \sum a_\alpha z^\alpha,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index of nonnegative integers and each a_α belongs to $\mathbb{C}^n$.

A mapping $F : \mathbb{B} \to \mathbb{B}$ is said to be bi-holomorphic if

- F is one-to-one and onto.
- F is holomorphic.
- F^{-1} is holomorphic.

The automorphism group of $\mathbb{B}$, denoted by $Aut(\mathbb{B})$, consists of all bi-holomorphic mappings of $\mathbb{B}$. It is clear that $Aut(\mathbb{B})$ is a group with composition being the group operation. Conventionally, bi-holomorphic mappings are also called automorphisms.

Let $H^\infty(\mathbb{B})$ be the set of all bounded holomorphic functions on $\mathbb{B}$. The function $||\cdot||_\infty : H^\infty(\mathbb{B}) \to \mathbb{R}_{\geq 0}$ such that

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{B}\}, \ \forall\ f \in H^\infty(\mathbb{B}), \tag{1.21}$$

is called the sup norm on $H^\infty(\mathbb{B})$, and $\|f\|_\infty$ is called the sup norm of f.

The sup norm on $H^\infty(\mathbb{B})$ has all the properties of a norm.

A Cauchy sequence in H^∞ is a sequence $\{f_n\}_{n\in\mathbb{N}}$, with the following property: for each $\epsilon > 0$ there exists an $n(\epsilon)$ such that

$$\|f_n - f_m\| \leq \epsilon, \ \forall\ n \geq n(\epsilon), \text{and } m \geq n(\epsilon). \tag{1.22}$$

Theorem 1.1. The algebra of bounded holomorphic functions with the sup norm is a Banach algebra.

1.4 The Space $\mathcal{L}^{p,\infty}$

Let ω be a measure space with a (positive) sigma-finite measure μ. The weak $\mathcal{L}^p$ space $\mathcal{L}^{p,\infty}(\mu)$, $0 < p < \infty$, consists of those measurable functions f on Ω for which

$$\|f\|_{p,\infty} := \sup_{0<\lambda<\infty} \lambda \cdot (\mu(f,\lambda))^{1/p} < \infty, \tag{1.23}$$

where $\mu(f,\lambda) = \mu\{x : |f(x)| > \lambda\} = \mu(\{x \in \Omega : |f(x)| > \lambda\})$.

Chebyshev's inequality,

$$\mu(g,\lambda) \leq \frac{1}{\lambda}\int_\Omega |g|\, d\mu,$$

shows that $\mathcal{L}^p \subset \mathcal{L}^{p,\infty}$, while the formula

$$\int_\Omega |g|^q\, d\mu = \int_0^\infty \mu(g,\lambda)\, d(\lambda^q) \tag{1.24}$$

(proved by means of Fubini's theorem) implies $\mathcal{L}^{p,\infty} \subset \mathcal{L}^q$ for $q < p$, if μ is finite. The quantity $\|\cdot\|_{p,\infty}$ is a norm for number p, but we have

$$\|f+g\|_{p,\infty} \leq C_p\left(\|f\|_{p,\infty} + \|f\|_{p,\infty}\right)\ \left(C_p = 2^{\max(1/p,1)}\right),$$

and hence $\|\cdot\|_{p,\infty}$ is a (complete) quasinorm. It is interesting, however, that if $p = 1$, then the space need not be locally convex (if, for example, $\Omega = [0,1]$ with Lebesgue measure), although it can be $q-$re-normed for every $q < 1$. For $p > 1$ the space is locally convex, and for $p < 1$ it is $p-$convex, i.e., there is an equivalent $p-$norm on it. Hence, the following inequalities holds

$$\mu(f_1+f_2, \lambda_1+\lambda_2) \leq \mu(f_1,\lambda_1) + (f_2,\lambda_2),$$

and

$$\mu(f_1 f_2, \lambda_1\lambda_2) \leq \mu(f_1,\lambda_1) + \mu(f_2,\lambda_2).$$

1.5 Smooth Manifold

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets, then $f : U \to V$ is called a diffeomorphism or $\mathcal{C}^\infty-$diffeomorphism if it is a $\mathcal{C}^\infty-$bijection and its inverse $f^{-1} : V \to U$ is $\mathcal{C}^\infty$ on V.

If $f : U \to V$ is a $\mathcal{C}^\infty-$diffeomorphism then for any $a \in U$ $D_a f : \mathbb{R}^n \to \mathbb{R}^m$ is an invertible linear map. In particular $n = m$.

We say that $f : U \to \mathbb{R}^n$ is a local diffeomorphism if any $x \in U$ has an open neighbourhood U_x such that

$$f : U_x \to f(U_x)$$

is a diffeomorphism. The inverse function theorem then implies that for any $a \in U$, $D_a f$ is invertible.

A pair $(M, \mathcal{A})$ where M is a topological space which is separable and second countable, and $\mathcal{A}$ is a collection of continuous maps $\{\phi_\alpha : U_\alpha \to M \ \forall \alpha \in I\}$, for open sets $U_\alpha \subset \mathbb{R}^n$ is a smooth manifold if the following conditions are satisfied:

(1) $\phi : U_\alpha \to \phi(U_\alpha)$ is a homeomorphism, and

$$\bigcup_\alpha \phi_\alpha(U_\alpha) = M,$$

(2) The charts (ϕ_α, U_α) are smoothly compatible i.e. for any $\alpha, \beta \in I$, with

$$\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta) \neq \emptyset,$$

$$\phi_\alpha^{-1} \circ \phi_\beta : \phi_\beta^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta)) \to \phi_\alpha^{-1}(\phi_\alpha(U_\alpha) \cap \phi_\beta(U_\beta))$$

is a diffeomorphism. The above condition makes $\mathcal{A}$ a smooth atlas.

(3) $\mathcal{A}$ is a maximal smooth atlas i.e. there is strictly no larger smooth atlas containing $\mathcal{A}$.

Definition 1.7. A Lie group is a group G which is a differentiable manifold and such that multiplication and inversion are smooth maps. By multiplication we mean that $(g, h) \mapsto gh : G \times G \to G$ is smooth and by inversion we mean that $g \mapsto g^{-1} : G \to G$ are smooth.

Let G and H be two Lie groups. Then $f : G \to H$ is a Lie group morphism if it is smooth and a group morphism. If $H \subset G$, then H is a Lie subgroup of G if it is at the same time a subgroup and submanifold of G.

A Lie algebra on a field $\mathbb{K}$ is a $\mathbb{K}-$vector space V endowed with an antisymmetric $\mathbb{K}-$bilinear map $[\cdot,\cdot] : V \times V \to V$ which satisfy the Jacobi identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

By antisymmetric we mean that

$$[x, y] = -[y, x]$$

and $[\cdot,\cdot]$ is called the Lie bracket.

A Lie subalgebra of V is a linear subspace of V which is a linear subspace of V which is stable under $[\cdot,\cdot]$.

A lie algebra morphism is a linear map A from $(V, [\cdot,\cdot]_V)$ to $(V', [\cdot,\cdot]_{V'})$ such that

$$[A(x), A(y)]_{V'} = A([x, y]_V).$$

A Lie algebra isomorphism is a bijective Lie algebra morphism, it is then automatic that its inverse is a Lie algebra morphism.

2. Algebra of Bergman Spaces

Here, we shall study the properties of the Bergman spaces and the Toeplitz operator algebra defined on them.

Definition 2.1. For $\alpha > 0$ the weighted Lebesgue measure $d\mathcal{A}_\alpha$ is defined by

$$dA_\alpha(z) = C_\alpha \left(1 - |z|^2\right)^\alpha dA(z)$$

where

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)},$$

is a normalizer so that $d\mathcal{A}_\alpha$ forms a probability measure on $\mathbb{B}_n$. For $\alpha > -1$ and $p > 0$ the weighted Bergman space $\mathcal{A}_\alpha^p$ consists of holomorphic functions f in $L^p(\mathbb{B}, d\mathcal{A})$, that is,

$$\mathcal{A}_p = L^p(\mathbb{B}, d\mathcal{A}) \cap H(\mathbb{B}).$$

It is clear that $\mathcal{A}_\alpha^p$ is a linear subspace of $L^p(\mathbb{B}, d\mathcal{A})$. When the weight $\alpha = 0$, we simply write $\mathcal{A}^p$ for $\mathcal{A}_\alpha^p$. These give the standard (unweighted) Bergman spaces.

The Bergman norm is a quasinorm given by

$$\|f\| = \|f\|_{\mathcal{A}^p} = \left(2\int_0^1 I_p(r,f)rdr\right)^{\frac{1}{p}}, \tag{2.1}$$

where

$$I_p(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{\frac{1}{p}}. \tag{2.2}$$

It is clear that $\mathcal{A}^p$ is also a linear subspace $L^p(\mathbb{B}, d\mathcal{A})$.

Proposition 2.1. *[15]* Let $p \in (0, \infty)$. Then, the following hold:

1. For every $z \in \mathbb{B}$ the functional $f \mapsto f(z)$ is continuous on $\mathcal{A}^p$;
2. The space $\mathcal{A}^p$ is complete;
3. If $f \in \mathcal{A}^p$, and $f_n(z) = f(nz)$, then $f_n \in \mathcal{A}^p$ and $\|f - f_n\| \to 0 (n \to \infty)$;
4. The set of all polynomials is a dense subset $\mathcal{A}^p$.

Proposition 2.2. The space $L^p(\mathbb{B}, d\mathcal{A})$ is a Banach algebra with convolution operation defined as

$$\|f * g\|^p = 2\int_0^1 I_p(r, f*g)\, rdr \tag{2.3}$$

for all $f, g \in L^p(\mathbb{B}, d\mathcal{A})$

Proof. Now,

$$\|f * g\|^p = 2\int_0^1 I_p(r, f*g)\, rdr.$$

To this end,

$$\begin{aligned} I_p(r, f*g) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f*g|^p \left(re^{i\theta}\right) d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x-y)dy\right|^p \left(re^{i\theta}\right) d\theta \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)g(x-y)dy|^p \left(re^{i\theta}\right) d\theta. \end{aligned}$$

Letting $\theta = x - y$ so that $x = \theta + y$ and $d\theta = dx$, we obtain

$$re^{i\theta} = re^{i(x-y)} = re^{ix-iy} = re^{ix} \cdot re^{-iy}.$$

Hence,

$$\begin{aligned}
\|f*g\|^p &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta+y)g(\theta)|^p\,dy\left(re^{ix}\right)\left(re^{-iy}\right)dx \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta+y)|^p\left(re^{-iy}\right)dy\,|g(z)|^p\left(re^{ix}\right)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta+y)|^p\,re^{-iy}dy\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\theta)|^p\left(re^{ix}\right)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta-y)|^p\,re^{iy}dy\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(\theta)|^p\left(re^{ix}\right)dx \\
&= I_p(r,f)\cdot I_p(r,g)
\end{aligned}$$

For this reason,

$$\|f*g\|_{\mathcal{A}^p} \le \|f\|_{\mathcal{A}^p}\,\|g\|_{\mathcal{A}^p}\,. \quad \square$$

Remark 2.3. In the special case when $p=2$, $L^2(\mathbb{B}, d\mathcal{A})$ is a Hilbert space whose inner product is denoted by

$$\langle f,g\rangle_{\mathcal{A}^p} = \int_{\mathbb{B}} f(z)\overline{g(z)}d\mathcal{A}(z) \tag{2.4}$$

and

$$\begin{aligned}
|\langle f,g\rangle|_{\mathcal{A}^p} &\le \left(\int_{\mathbb{B}}|f(z)|^2\,d\mathcal{A}(z)\right)^{1/2}\left(\int_{\mathbb{B}}|g(z)|^2\,d\mathcal{A}(z)\right)^{1/2} \\
&= \|f\|_{\mathcal{A}^p}\,\|g\|_{\mathcal{A}^p}
\end{aligned}$$

Theorem 2.4. Suppose x and y are complex numbers, φ and ϕ are bounded functions on $\mathbb{B}$; then

1. $T_{x\varphi} + T_{y\phi} = xT_\varphi + yT_\phi$;
2. $T_{\overline{\varphi}} = T_\varphi^*$;
3. $T_\varphi \ge 0$ if $\varphi \ge 0$;
   Moreover, if $\varphi \in H^\infty$, then
4. $T_\phi T_\varphi = T_{\phi\varphi}$;
5. $T_{\overline{\varphi}} T_\phi = T_{\overline{\varphi}\phi}$.

Proof: Given that, $\varphi, \phi \in L^\infty(\mathbb{B})$, then for any $x, y \in \mathbb{C}$, (1) implies

$$\begin{aligned}
T_{x\varphi}(f) + T_{y\varphi}(f) &= P(x\varphi f) + P(y\phi f) \\
&= \int_{\mathbb{B}} K(z,u)(x\varphi)(u)f(u)d\mathcal{A}(u) + \int_{\mathbb{B}} K(z,u)(y\varphi)(u)f(u)d\mathcal{A}(u) \\
&= \int_{\mathbb{B}} \frac{(x\varphi)(u)f(u)}{(1-z\overline{u})^2}d\mathcal{A}(u) + \int_{\mathbb{B}} \frac{(y\varphi)(u)f(u)}{(1-z\overline{u})^2}d\mathcal{A}(u) \\
&= \int_{\mathbb{B}} \frac{x\varphi(u)f(u)}{(1-z\overline{u})^2}d\mathcal{A}(u) + \int_{\mathbb{B}} \frac{y\varphi(u)f(u)}{(1-z\overline{u})^2}d\mathcal{A}(u) \\
&= x\int_{\mathbb{B}} \frac{\varphi(u)f(u)}{(1-z\overline{u})^2}d\mathcal{A}(u) + y\int_{\mathbb{B}} \frac{\varphi(u)f(u)}{(1-z\overline{u})^2}d\mathcal{A}(u) \\
&= x\int_{\mathbb{B}} K(z,u)\varphi(u)f(u)d\mathcal{A}(u) + y\int_{\mathbb{B}} K(z,u)\varphi(u)f(u)d\mathcal{A}(u) \\
&= xP(\varphi f) + yP(\phi f) \\
&= xT_\varphi(f) + yT_\phi(f).
\end{aligned}$$

(2) implies;

$$\begin{aligned}
T_{\overline{\varphi}}(f) &= P(\overline{\varphi} f)\\
&= \int_{\mathbb{B}} K(z,u)\overline{\varphi}(u) f(u) d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} \frac{\overline{\varphi}(u) f(u)}{(1-z\overline{u})^2} d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} \frac{(\varphi(u) f(u))^*}{(1-z\overline{u})^2} d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} K(z,u)\left(\varphi(u) f(u)\right)^* d\mathcal{A}(u)\\
&= P(\varphi f)^*\\
&= T^*_{\varphi}(f).
\end{aligned}$$

(3.) Since there are non zero divisors in the set of all Toeplitz operators for all $\varphi \in L^\infty(\mathbb{B})$, then

$$T_\varphi(f) = 0$$

implies

$$\begin{aligned}
T_\varphi(f) &= P(\varphi f)(z)\\
&= \int_{\mathbb{B}} K(z,u)\varphi(u) f(u) d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} \frac{\varphi(u) f(u)}{(1-z\overline{u})^2} d\mathcal{A}(u)\\
&= 0.
\end{aligned}$$

Since $f(u) \neq 0$, $\varphi(u)$ must be equal to zero. Also, $T_\varphi(f) > 0$ implies

$$T_\varphi(f) = P(\varphi f) > 0 \implies \varphi > 0.$$

Hence,

$$T_\varphi(f) \geq 0 \implies P(\varphi f) \geq 0 \implies \varphi \geq 0.$$

(4) If $\varphi, \phi \in H^\infty$, then $\varphi \mathcal{A}^2 \subseteq \mathcal{A}^2$ and,

$$\begin{aligned}
T_\phi T_\varphi(f) &= P(\phi f) P(\varphi f)\\
&= \left\langle \int_{\mathbb{B}} K(z,u)\phi(u) f(u) d\mathcal{A}(u), \int_{\mathbb{B}} K(z,u)\varphi(u) f(u) d\mathcal{A}(u)\right\rangle\\
&= \int_{\mathbb{B}} K(z,u)\left\langle \phi, \varphi\right\rangle f(u) d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} \frac{\left\langle \phi, \varphi\right\rangle f(u)}{(1-z\overline{u})^2} d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} \frac{\phi\varphi f(u)}{(1-z\overline{u})^2} d\mathcal{A}(u)\\
&= \int_{\mathbb{B}} K(z,u)\phi(u)\varphi(u) f(u) d\mathcal{A}(u)\\
&= P(\phi\varphi f)\\
&= T_{\phi\varphi}(f).
\end{aligned}$$

(5) follows from (2) and (4) that is, by taking the adjoint,

$$\begin{aligned}
T_{\overline{\varphi}}(f) T_\phi(f) &= T^*_\varphi(f) T^*_{\overline{\phi}}(f)\\
&= \left(T_\varphi(f) T_{\overline{\phi}}(f)\right)^*\\
&= \left(T_{\varphi\overline{\phi}}(f)\right)^*\\
&= T_{\overline{\varphi}\phi}(f)
\end{aligned}$$

and therefore, the proof is complete. □

3. Algebra of Bloch Spaces

Definition 3.1. For $0 \leq \alpha < \infty$, let $\mathcal{H}_{\alpha}^{\infty}$ be the space of holomorphic functions $f \in \mathcal{H}(\mathbb{B})$ satisfying

$$\sup_{z\in\mathbb{B}} \left(1 - \|z\|^2\right)^{\alpha} |f(z)| < \infty. \tag{3.1}$$

We abbreviate $\mathcal{H}^{\infty} = \mathcal{H}_1^{\infty}$ for $\alpha = 1$.

The classical $\alpha-$Bloch space, denoted as $\mathcal{B}^{\alpha}$ is the space of holomorphic functions $F : \mathbb{B} \to \mathbb{C}$ satisfying

$$\|f\|_{\mathcal{B}^{\alpha}(\mathbb{B})} = \sup_{z\in\mathbb{B}} \left(1 - \|z\|^2\right)^{\alpha} |f'(z)| < +\infty. \tag{3.2}$$

Now we introduce four semi-norms of the Bloch type space (see [18]) for $f \in \mathcal{H}(\mathbb{B})$ in what follows. To do this, let

$$\|f\|_{\mathcal{B},\alpha} := \sup_{z\in\mathbb{B}} \left(1 - \|z\|^2\right)^{\alpha} |\partial^m f(z)|\,, \tag{3.3}$$

$$\|f\|_{\mathcal{R},\alpha} := \sup_{z\in\mathbb{B}} \left(1 - \|z\|^2\right)^{\alpha} |\mathcal{R}f(z)|\,, \tag{3.4}$$

$$\|f\|_{weak,\alpha} := \sup_{z\in\mathbb{T}} \|f_y\|_{\mathcal{B}^{\alpha}(\mathbb{B})}\,, \tag{3.5}$$

where $\mathcal{R}f(z) = \langle \partial^m f(z), z\rangle\,,\ f_y(z) = f(zy)$ for $z \in \mathbb{B},\ |z| < 1$, for each $y \in \mathbb{B}$ with norm $||y|| = 1$. We note that $zf_y'(z) = \mathcal{R}f(zy)$. The Mobius transforms of $\mathbb{B}$ are holomorphic mappings $\varphi_a,\ a \in \mathbb{B}$, given by

$$\varphi_a(z) = (P_a + s_a Q_a)(m_a(z))\,, \tag{3.6}$$

where $s_a = \sqrt{1 - \|a\|^2},\ P_a(z) = \frac{\langle z,a\rangle}{\langle a,a\rangle}a,\ Q_a = I - P_a$ and $m_a(z) = \frac{a-z}{1-\langle z,a\rangle}$. Define

$$\|f\|_{\tilde{\mathcal{B}},\alpha} := \sup_{z\in\mathbb{B}} \left(1 - \|z\|^2\right)^{\alpha-1} \left|\tilde{\nabla} f(z)\right|, \tag{3.7}$$

where $\tilde{\nabla} = \partial^m f \circ \varphi_z(0)$ with $\varphi_z \in \mathrm{Aut}(\mathbb{B})$. We note that, by Lemma 3.5 of [1]

$$\left\|\tilde{\nabla} f(z)\right\| = \sup_{w\neq 0} \frac{(1 - ||z||^2)\,|\partial^m f(z)(w)|}{\sqrt{(1 - ||z||^2)||w||^2 + |\langle w, z\rangle|^2}}. \tag{3.8}$$

Hence, we have

$$\|f\|_{\tilde{\mathcal{B}},\alpha} = \sup_{x\in\mathbb{B}}\sup_{w\neq 0} \frac{(1 - ||z||^2)^{\alpha}\,|\partial^m f(z)(w)|}{\sqrt{(1 - ||z||^2)||w||^2 + |\langle w, z\rangle|^2}}. \tag{3.9}$$

Hence for $\alpha > 0$, we have

$$\mathcal{B}^{\alpha} = \left\{f \in \mathcal{H}(\mathbb{B}) : \|f\|_{\tilde{\mathcal{B}},\alpha} < +\infty\right\}. \tag{3.10}$$

Proposition 3.1. Equipped with the norm $\|f\|_{\alpha} = |f(0)| + \|f\|_{\mathcal{B},\alpha}$ for $f \in \mathcal{B}^{\alpha}$, the Bloch type space $\mathcal{B}^{\alpha}$ becomes a Banach space and hence, a Banach algebra.

Proof. For any $f, g \in \mathcal{B}^{\alpha}$ we have,

$$\begin{aligned}\|fg\|_{\alpha} &= \|fg(0)\| + \sup_{z\in\mathbb{B}}(1 - ||z||^2)^{\alpha}\,\|(fg)'(z)\|_{\alpha}\\ &= \|fg(0)\| + \sup_{z\in\mathbb{B}}(1 - ||z||^2)^{\alpha}\,\|(f'g)(z) + (fg'(z))\|_{\alpha}\\ &\leq \|f(0)\|\,\|g(0)\| + \sup_{z\in\mathbb{B}}(1 - ||z||^2)^{\alpha}\,(\|f'g(z)\|_{\alpha} + \|fg'(z)\|_{\alpha})\end{aligned}$$

$$
\begin{aligned}
&= \|f(0)\| \, \|g(0)\| + \sup_{z\in\mathbb{B}}(1-||z||^2)^\alpha \, \|f'g(z)\|_\alpha + \sup_{z\in\mathbb{B}}(1-||z||^2)^\alpha \, \|fg'(z)\|_\alpha \\
&\le \|f(0)\| \, \|g(0)\| + \sup_{z\in\mathbb{B}}(1-||z||^2)^\alpha \, \|f'(z)\|_\alpha \, \|g(z)\|_\alpha \\
&+ \sup_{z\in\mathbb{B}}(1-||z||^2)^\alpha \, \|f(z)\|_\alpha \, \|g'(z)\|_\alpha \\
&= |f(0)||g(0)| + \|f'(z)\|_\alpha \, \|g(z)\|_\alpha + \|f(z)\|_\alpha \, \|g'(z)\|_\alpha . \qquad \square
\end{aligned}
$$

Definition 3.2 (The Little Bloch Space)**.** We denote the class of Bloch functions defined on $\mathbb{B}$ by $\mathcal{B}(\mathbb{B})$. The Bloch space is not separable but there exists a separable subspace of the Bloch space known as the little Bloch space.

We let $\mathcal{C}(\overline{\mathbb{B}_n})$ be the space of continuous function on the closed unit ball, and $\mathcal{C}_0(\mathbb{B}_n)$ be closed subspace of $\mathcal{C}(\overline{\mathbb{B}_n})$ consisting of those functions that vanish on the boundary $\mathbb{T}_n$.

The little Bloch space $\mathcal{B}_0(\mathbb{B})$ is the subspace of $\mathcal{B}(\mathbb{B})$ given by those functions for which

$$\lim_{|z|\to 1-} \left|\tilde{\nabla} f(z)\right| = 0.$$

Since $\left|\tilde{\nabla} f(z)\right|$ is dense in $\mathbb{B}_n$, the above condition means that the function $\left|\tilde{\nabla} f(z)\right|$ belongs to $\mathcal{C}_0(\mathbb{B}_n)$.

Now, if $f \in \mathcal{B}(\mathbb{B})$ then $x^* f \in \mathcal{B}$ for all $x^* \in \mathbb{B}$. And, interchanging the suprema, we have that

$$\|f\|_{\mathcal{B}(\mathbb{B})} \approx \sup_{\|x^*\|=1} \|x^* f\|_{\mathcal{B}}$$

where $x^* f(z) = \langle f(z), x\rangle$.

The Bloch space possesses the following properties.

Definition 3.3 (Pointwise Multiplier on Bloch Spaces)**.** From definition 1.8 number (5), a function f is called a pointwise multiplier of a space $\mathcal{B}(\mathbb{B})$ if for every $g \in \mathcal{B}(\mathbb{B})$ the pointwise product fg also belongs to $\mathcal{B}(\mathbb{B})$. Thus, we denote a pointwise multiplier f of a space $\mathcal{B}(\mathbb{B})$ by $f\mathcal{B}(\mathbb{B}) \subset \mathcal{B}(\mathbb{B})$.

Here, we consider the pointwise multiplier algebra of the Bloch space. Throughout this section we use the following norm on $\mathcal{B}$:

$$\|g\| = |g(0)| + \sup\left\{\left(1-|z|^2\right)|\nabla g(z)| : z \in \mathbb{B}_n\right\}, \; g \in \mathcal{B}.$$

Proposition 3.2. For all $f, g \in \mathcal{B}(\mathbb{B})$ and $\varphi \in H^\infty(\mathbb{B})$, the following properties hold.

(i). $\|\mathcal{M}_{a\varphi}(f)\| = |a| \, \|\mathcal{M}_\varphi(f)\|$,

(ii) $\|\mathcal{M}_{a\varphi}(f)\| + \|\mathcal{M}_{b\varphi}(f)\| = |a| \, \|\mathcal{M}_\varphi(f)\| + |b| \, \|\mathcal{M}_\varphi(f)\|$ and

(iii). $\|\mathcal{M}_{\varphi_1\varphi_2}(f)\| \le \|\varphi_1\|_\infty \, \|\varphi_2\|_\infty$,

for all $\varphi_1, \varphi_2 \in H^\infty$ and a, b are constants. In particular, $\mathcal{M}_\varphi$ is a Banach algebra.

Proof. Given that

$$\|f(z)\| = |f(0)| + \|f(z)\|$$

where $\|f(z)\| = \sup_{z\in\mathbb{B}} (1+|z|^2)\,|\nabla f(z)|$ for $f \in \mathcal{B}$, then (i) implies

$$
\begin{aligned}
\|\mathcal{M}_{a\varphi}(f)\| &= |(a\varphi)f(0)| + \|(a\varphi)f(z)\| \\
&\le |a| \, \|\varphi f(0)\| + |a| \, \|\varphi f(z)\| \\
&\le |a| \, \|\varphi\|_\infty \, (|f(0)| + \|f(z)\|) \\
&= |a| \, \|\varphi\|_\infty \, \|f(z)\|_{\mathcal{B}} \\
& |a| \, \|\mathcal{M}_\varphi(f)\| .
\end{aligned}
$$

(ii) follows immediately from (i). And now (iii) can be shown by applying the close graph theorem as follows;

$$\begin{aligned}\|\mathcal{M}_{\varphi_1\varphi_2}(f)\| &= |(\varphi_1\varphi_2 f(0))| + \|\varphi_1\varphi_2 f(z)\| \\ &\leq \|\varphi_1\varphi_2\| \left(|f(0)| + \|f(z)\|\right) \\ &\leq \|\varphi_1\varphi_2\|_\infty \|f(z)\|_{\mathcal{B}} \\ &\leq \sup_{f\in\mathcal{B}} \frac{\|\varphi_1\varphi_2\| \|f(z)\|_{\mathcal{B}}}{\|f(z)\|_{\mathcal{B}}} \\ &= \|\varphi_1\varphi_2\|_\infty \\ &\leq \|\varphi_1\|_\infty \|\varphi_2\|_\infty . \quad \square\end{aligned}$$

4. Algebra of Hardy Spaces

Definition 4.1. For $0 < p < \infty$ and T_n a sphere of radius n which is the boundary of $\mathbb{B}_n$, the Hardy space H^p consists of holomorphic functions f in $\mathbb{B}_n$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_{\mathbb{T}_n} \left|f(re^{i\theta})\right|^p dm(\theta) < \infty. \tag{4.1}$$

Just as the Bergman kernel plays an essential role in the study of Bergman spaces, two integral kernels are fundamental in the theory of Hardy spaces; they are the Cauchy-Szego kernel,

$$\mathbf{C}_s(z,\xi) = \frac{1}{(1-\langle z,\xi\rangle^2)^n}, \tag{4.2}$$

and the (invariant) Poisson kernel,

$$P(z,\xi) = \frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}. \tag{4.3}$$

We note that, the Poisson kernel here is different from the associated Poisson kernel when $\mathbb{B}_n$ is thought of as the unit ball in $\mathbb{R}^{2n}$, unless $n = 1$. Hence, if f belongs to the ball algebra, then

$$f(z) = \int_{\mathbb{T}_n} \mathbf{C}_s(z,\xi) f(\xi) d\mu(\xi) \tag{4.4}$$

$$f(z) = \int_{\mathbb{T}_n} P(z,\xi) f(\xi) d\mu(\xi) \tag{4.5}$$

$$|f(z)|^p \leq \int_{\mathbb{T}_n} P(z,\xi) |f(\xi)|^p d\mu(\xi) \tag{4.6}$$

for all $z \in \mathbb{B}_n$ and $0 < p < \infty$.

Theorem 4.1. If f and g are bounded on $\mathbb{B}$ and a, b are complex numbers, then

1. $T_{af+bg} = aT_f + bT_g$;
2. $T_{\bar{f}} = T_f^*$, where T^* is the adjoint of T;

   If further f is in H^∞, then

3. $T_g T_f = T_{gf}$;
4. $T_{\bar{f}} T_g = T_{\bar{f}g}$

Proof: These properties follow easily from the definition of Toeplitz operators and simple calculations with the inner product in H^2 shows that: for all $f, g \in H^2$ and $a, b \in \mathbb{B}$, (1) implies

$$\begin{aligned}
T_{af} + T_{bg} &= \mathbf{C}_s(af) + \mathbf{C}_s(bg) \\
&= \int_{\mathbb{B}} \frac{af(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) + \int_{\mathbb{B}} \frac{bg(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= a\int_{\mathbb{B}} \frac{f(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) + b\int_{\mathbb{B}} \frac{g(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= a\mathbf{C}_s(f) + b\mathbf{C}_s(g) \\
&= aT_f + bT_g.
\end{aligned}$$

(2) $\implies$

$$\begin{aligned}
T_{\overline{f}} &= \mathbf{C}_s(\overline{f}) \\
&= \int_{\mathbb{B}} \frac{\overline{f}(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= \int_{\mathbb{B}} \frac{f^*(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= \mathbf{C}_s(f)^* \\
&= T_f^*.
\end{aligned}$$

Now, given that $f, g \in H^\infty$, then (3) implies

$$\begin{aligned}
T_g T_f &= \mathbf{C}_s(g)\mathbf{C}_s(f) \\
&= \int_{\mathbb{B}} \frac{g(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \int_{\mathbb{B}} \frac{f(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= \int_{\mathbb{B}} \frac{1}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \int_{\mathbb{B}} \frac{g(\xi)f(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= \mathbf{C}_1 \int_{\mathbb{B}} \frac{f(\xi)g(\xi)}{(1-\langle z,\xi\rangle)^n} d\beta(\xi) \\
&= \mathbf{C}_{s_1}\mathbf{C}_{s_2}(gf) \\
&= \mathbf{C}_s(gf) \quad (\text{taken } \mathbf{C}_{s_1}\mathbf{C}_{s_2} = \mathbf{C}_s) \\
&= T_{gf}.
\end{aligned}$$

Combining (2) and (3), (4) implies

$$\begin{aligned}
T_{\overline{f}} T_g &= \mathbf{C}_s(\overline{f})\mathbf{C}_s(g) \\
&= \mathbf{C}_s(\overline{f}g) \\
&= T_{\overline{f}g}. \quad \square
\end{aligned}$$

5. Algebra of Spherical Functions

In this section, we consider the algebra of spherical functions. They are the Gelfand pair analogue of characters on locally compact Abelian groups. Let G be a locally compact group and K be a compact subgroup. In the foremost, we do not require (G, K) to be a Gelfand pair.

Definition 5.1. *[17],[16]* A **spherical measure** for (G, K) is a non zero Radon measure μ on G such that

(1). μ is K bi-invariant i.e. $\mu(k_1 E k_2^{-1}) = \mu(E)$ for every Borel set $E \subset G$, and

(2). $f \mapsto \mu(f) = \int_G f(g)d\mu(g)$ is an algebra homomorphism $C_c(K \setminus G/K) \to \mathbb{C}$, i.e.

$$\mu(f_1 * f_2) = \mu(f_1)\mu(f_2). \tag{5.1}$$

In other words, μ is multiplicative linear functional on $C_c(K \setminus G/K)$.

Citation: Egwe, M. E. (2025). A Note on Some Classes of Function Algebras. *J Theor Phys Math, 3*(2), 1-24.

Definition 5.2. *[8]* A **Spherical function** for (G,K) is a continuous function $\phi : G \to \mathbb{C}$ such that the integral $\phi(f)$ defined by

$$\phi(f) = \int_G f(x)\phi(x^{-1})d\mu_G(x) \tag{5.2}$$

where μ is a spherical measure on (G,K), is spherical for (G,K). Then it is automatic that ϕ is bi-K-invariant and that $\phi(1) = 1$.

Remark 5.1. By a representation $F : L^1(K \setminus G/K) \to \mathbb{C}$ of G on a Banach space $\mathcal{H} = L^2(K \setminus G/K)$ we define the homomorphism

$$f \mapsto F_g(f) = \int_G f(g)\phi(g^{-1})d\mu_G(g)$$

of G into the group of non-singular bounded operators $\mathcal{B}(\mathcal{H})$, with the requirement that, for every $f \in \mathcal{H}$, the mapping

$$f \mapsto F_g(f)$$

of G into $\mathcal{H}$ is (strongly) continuous. For every compact set $K \subset G$ and every $f \in \mathcal{H}$ the function $F_g(f)$, $g \in K$, is also compact and bounded in $\mathcal{H}$ by (Banach-Steinhaus theorem) this shows that

$$\phi(g) = \|F_g(f)\| \tag{5.3}$$

is bounded on every compact set, where $\phi(g)$ is lower semi-continuous on G, and satisfies

$$\phi(g_1g_2) \le \phi(g_1)\phi(g_2) \tag{5.4}$$

for every two $g_1, g_2 \in K$; such a function will be called a **semi-norm** on G.

Proposition 5.2. Given a semi-norm ϕ and for every $f \in L^1(K \setminus G/K)$, with

$$\|f\|_\phi = \int_G |f(g)|\,\phi(g)d\mu_G(g), \tag{5.5}$$

$L^1(K \setminus G/K)$ is a topological algebra.

Proof: For all $f_1, f_2 \in L^1(K \setminus G/K)$, (5.5) implies

$$\begin{aligned}
\|F_g(f_1 * f_2)(\phi)\| &\le \|F_g\|\,\|f_1 * f_2(\phi)\| \\
&= \phi(g)\int_G |f_1 * f_2(g)|\,d\mu(g) \\
&= \phi(g)\int_G\int_G \left|f_1(g)f_2(y^{-1}g)\right| d\mu(g)d\mu(y) \\
&\le \phi(g)\int_G\int_G |f_1(g)|\left|f_2(y^{-1}g)\right| d\mu(g)d\mu(y) \\
&= \phi(g)\,\|f_1(g)\|\int_G \left|f_2(y^{-1}g)\right| d\mu(y) \\
&= \phi(g)\,\|f_1\|\,\|f_2\|\,.
\end{aligned}$$

Hence,

$$\|f_1 * f_2\|_\phi \le \|f_1\|_\phi\,\|f_2\|_\phi\,. \tag{5.6}$$

To this end, the space $C_c(K \setminus G/K)$ of absolutely integrable functions with respect to $\phi(g)d\mu_G(g)$ can be considered in a natural way as a complete normed algebra under the convolution product. Of course, $L^1(G)$ is everywhere dense in $C_c(K \setminus G/K)$ by the definition of integrable functions.

The norm

$$\|f\| = \int_G |f(x)|\,dx$$

turns the group algebra into a normed vector space. Owing to the additional property

$$\|f_1 * f_2\| \leq \|f_1\| \, \|f_2\| \, ,$$

The algebra $C_c^\sharp(G)$ is a closed subalgebra. □

6. Algebra of Lie Pseudogroups

Here, we shall follow [19] introduce Lie pseudogroups and their Lie algebras.

Definition 6.1. Let M be a smooth manifold and let $\mathcal{G}$ be a collection of diffeomorphisms of open subsets of M into M. $\mathcal{G}$ is called a pseudogroup if the following hold:

(1) $\mathcal{G}$ is closed under restriction: if $f \in \mathcal{G}$ and U is a domain of f, then $f|_V \in \mathcal{G}$ for any open $V \subset U$.

(2) if $f : U \to M$ is a diffeomorphism, $U = \bigcup\limits_{\alpha} U_\alpha$, and $f|_{U_\alpha} \in \mathcal{G}$, then $f \in \mathcal{G}$.

(3) $\mathcal{G}$ is closed under inverse: if $f \in \mathcal{G}$, then $f^{-1} \in \mathcal{G}$.

(4) $\mathcal{G}$ is closed under composition: $f : U \to M$ and $g : f(U) \to M$ both belong to $\mathcal{G}$, then $g \circ f \in \mathcal{G}$.

(5) The identity diffeomorphism $M \to M$ belongs to $\mathcal{G}$.

By $\mathcal{J}^n(M)$ we denote the manifold of all $n-$jets of all diffeomorphisms of open subsets of M into M. By $\mathcal{J}^n\mathcal{G}$ we denote the set of all $n-$jets of all diffeomorphisms belonging to $\mathcal{G}$.

Definition 6.2. A pseudogroup $\mathcal{G}$ is a Lie pseudogroup if there exists an integer $n \geq 0$, called the order of $\mathcal{G}$, such that

(1) The set $\mathcal{J}^n\mathcal{G}$ is a smooth submanifold of $\mathcal{J}^n(M)$.

(2) A diffeomorphism $f : U \to M$ belongs to $\mathcal{G}$ if and only if $[f]_p^n \in \mathcal{J}^n\mathcal{G}$ for all $p \in U$.

The submanifold $\mathcal{J}^n\mathcal{G}$ of a Lie pseudogroup $\mathcal{G}$ is called a system of Partial Differential Equations defining $\mathcal{G}$.

A pseudogroup $\mathcal{G}$ is transitive if for any p_1, p_2 in M there exists $f \in \mathcal{G}$ such that $f(p_1) = p_2$.

6.1 Lie Algebra of Pseudogroups

Let $\mathcal{G}$ be a Lie pseudogroup acting on manifold M, β a vector field in M, and let φ_t be the flow of β. The vector field β is $\mathcal{G}-$vector field if its flow consists of diffeomorphisms belonging to $\mathcal{G}$, that is $\beta \in \mathcal{G}$ for all t.

Proposition 6.1. The set of all $\mathcal{G}-$vector fields is a Lie subalgebra in the Lie algebra of all vector fields in M.

Thus, the following hold:

(1) Suppose β_1 and β_2 are vector fields in M. Then

$$[\beta_1^n, \beta_2^n] = [\beta_1, \beta_2]^n \, .$$

(2) A vector field β_1 in M is a $\mathcal{G}-$vector field if and only if the vector field β^n is tangent to the equation $\mathcal{J}^n\mathcal{G}$.

The statement of the proposition follows from these facts.

The Lie algebra of all $\mathcal{G}-$vector fields is called the Lie algebra of $\mathcal{G}$. We denote it by $\mathfrak{G}$.

Proof: The proof can be would have been completed if we can establish the following lemmas.

Lemma 6.2. Suppose β_1 and β_2 are vector fields in M. Then

$$[\beta_1^n, \beta_2^n] = [\beta_1, \beta_2]^n .$$

Proof. Let φ_{1t} be the flow of the vector field β_1 in M and let φ_{2t} be the flow of the vector field β_2 in M. Then the flow φ_t^n is defined in $\mathcal{J}^n(M)$ by the formula

$$\beta_1^n = \varphi_{1t}\left([f]_{p_1}^n\right) = [\varphi_{1t}]_{f(p_1)}^n \cdot [f]_{p_1}^n = [\varphi_{1t} \circ f]_{p_1}^n$$

and

$$\beta_2^n = \varphi_{2t}\left([f]_{p_2}^n\right) = [\varphi_{2t}]_{f(p_2)}^n \cdot [f]_{p_2}^n = [\varphi_{2t} \circ f]_{p_2}^n ,$$

then by induction when $n = 1$, we have

$$\begin{aligned}
[\beta_1, \beta_2] &= \left[\varphi_{1t}\left([f]_{p_1}\right), \varphi_{2t}\left([f]_{p_2}\right)\right] \\
&= \left[[\varphi_{1t}]_{f(p_1)} \cdot [f]_{p_1}, [\varphi_{2t}]_{f(p_2)} \cdot [f]_{p_2}\right] \\
&= \left[[\varphi_{1t} \circ f]_{p_1}, [\varphi_{2t} \circ f]_{p_2}\right] \\
&= [\varphi_{1t} \circ f]_{p_1} [\varphi_{2t} \circ f]_{p_2} - [\varphi_{2t} \circ f]_{p_2} [\varphi_{1t} \circ f]_{p_1} \\
&= \beta_1\beta_2 - \beta_2\beta_1.
\end{aligned}$$

When $n = 2$, we have

$$\begin{aligned}
[\beta_1^2, \beta_2^2] &= \left[\varphi_{1t}\left([f]_{p_1}^2\right), \varphi_{2t}\left([f]_{p_2}^2\right)\right] \\
&= \left[[\varphi_{1t}]_{f(p_1)}^2 \cdot [f]_{p_1}^2, [\varphi_{2t}]_{f(p_2)}^2 \cdot [f]_{p_2}^2\right] \\
&= \left[[\varphi_{1t} \circ f]_{p_1}^2, [\varphi_{2t} \circ f]_{p_2}^2\right] \\
&= [\varphi_{1t} \circ f]_{p_1}^2 [\varphi_{2t} \circ f]_{p_2}^2 - [\varphi_{2t} \circ f]_{p_2}^2 [\varphi_{1t} \circ f]_{p_1}^2 \\
&= \left([\varphi_{1t} \circ f]_{p_1} [\varphi_{2t} \circ f]_{p_2} - [\varphi_{2t} \circ f]_{p_2} [\varphi_{1t} \circ f]_{p_1}\right)^2 \\
&= (\beta_1\beta_2 - \beta_2\beta_1)^2 \\
&= [\beta_1, \beta_2]^2 .
\end{aligned}$$

Now, when $n = n + 1$ we have

$$\begin{aligned}
[\beta_1^{n+1}, \beta_2^{n+1}] &= \left[\varphi_{1t}\left([f]_{p_1}^{n+1}\right), \varphi_{2t}\left([f]_{p_2}^{n+1}\right)\right] \\
&= \left[[\varphi_{1t}]_{f(p_1)}^{n+1} \cdot [f]_{p_1}^{n+1}, [\varphi_{2t}]_{f(p_2)}^{n+1} \cdot [f]_{p_2}^{n+1}\right] \\
&= \left[[\varphi_{1t} \circ f]_{p_1}^{n+1}, [\varphi_{2t} \circ f]_{p_2}^{n+1}\right] \\
&= [\varphi_{1t} \circ f]_{p_1}^{n+1} [\varphi_{2t} \circ f]_{p_2}^{n+1} - [\varphi_{2t} \circ f]_{p_2}^{n+1} [\varphi_{1t} \circ f]_{p_1}^{n+1} \\
&= \left([\varphi_{1t} \circ f]_{p_1} [\varphi_{2t} \circ f]_{p_2}\right)^{n+1} - \left([\varphi_{2t} \circ f]_{p_2} [\varphi_{1t} \circ f]_{p_1}\right)^{n+1} \\
&= (\beta_1\beta_2)^{n+1} - (\beta_2\beta_1)^{n+1} \\
&= [\beta_1, \beta_2]^{n+1} .
\end{aligned}$$

Since this is true for $n = 1$, $n = 2$ and $n = n + 1$, it is therefore true for all n. □

The antisymmetric property of the Lie algebra follows directly from the following:

Lemma 6.3. Suppose β, β_1, β_2 are vector fields in M, for all $p, p_1, p_2 \in M$, then

(1) $[\beta, \beta] = 0$ and

(2) $[\beta_1, \beta_2] = -[\beta_2, \beta_1]$.

Proof. (1). Let φ_t be the flow of β^n in M with φ_t defined in $\mathcal{J}^n(M)$ as

$$\varphi_t\left([f]_p^n\right) = [\varphi_t]_{f(p)}^n \cdot [f]_p^n = [\varphi_t \circ f]_p^n$$

for all $p \in M$. Then $[\beta^n, \beta^n]$ implies

$$\begin{aligned}
\left[\varphi_t\left([f]_p^n\right), \varphi_t\left([f]_p^n\right)\right] &= \left[[\varphi_t \circ f]_p^n, [\varphi_t \circ f]_p^n\right] \\
&= [\varphi_t \circ f]_p^n [\varphi_t \circ f]_p^n - [\varphi_t \circ f]_p^n [\varphi_t \circ f]_p^n \\
&= [\varphi_t]_{f(p)}^n \cdot [f]_p^n [\varphi_t]_{f(p)}^n \cdot [f]_p^n - [\varphi_t]_{f(p)}^n \cdot [f]_p^n [\varphi_t]_{f(p)}^n \cdot [f]_p^n \\
&= [\varphi_t]_{f(p)}^{2n} [f]_p^{2n} - [\varphi_t]_{f(p)}^{2n} [f]_p^{2n} \\
&= [\varphi_t]_{f(p)}^{2n} \left([f]_p^{2n} - [f]_p^{2n}\right) = [\varphi_t]_{f(p)}^{2n} \cdot 0 = 0.
\end{aligned}$$

(2). We have

$$\begin{aligned}
&\left[\varphi_{1t}\left([f]_{p_1}^n\right), \varphi_{2t}\left([f]_{p_2}^n\right)\right] \\
&= \left[[\varphi_{1t} \circ f]_{p_1}^n, [\varphi_{2t} \circ f]_{p_2}^n\right] \\
&= [\varphi_{1t} \circ f]_{p_1}^n [\varphi_{2t} \circ f]_{p_2}^n - [\varphi_{2t} \circ f]_{p_2}^n [\varphi_{1t} \circ f]_{p_1}^n \\
&= \left([\varphi_{1t} \circ f]_{p_1} [\varphi_{2t} \circ f]_{p_2} - [\varphi_{2t} \circ f]_{p_2} [\varphi_{1t} \circ f]_{p_1}\right)^n \\
&= [\varphi_{1t}]_{f(p_1)}^n [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n - [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n [\varphi_{1t}]_{f(p_1)}^n [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n \\
&= -[\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n [\varphi_{1t}]_{f(p_1)}^n [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n + [\varphi_{1t}]_{f(p_1)}^n [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n \\
&= -\left([\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n [\varphi_{1t}]_{f(p_1)}^n [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n - [\varphi_{1t}]_{f(p_1)}^n [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n [f]_{p_2}^n\right) \\
&= -\left([\varphi_{2t} \circ f]_{p_2}^n [\varphi_{1t} \circ f]_{p_1}^n - [\varphi_{1t} \circ f]_{p_1}^n [\varphi_{2t} \circ f]_{p_2}^n\right) \\
&= -\left(\varphi_{2t}\left([f]_{p_2}^n\right) \varphi_{1t}\left([f]_{p_1}^n\right) - \varphi_{1t}\left([f]_{p_1}^n\right) \varphi_{2t}\left([f]_{p_2}^n\right)\right) \\
&= -\left[\varphi_{2t}\left([f]_{p_2}^n\right), \varphi_{1t}\left([f]_{p_1}^n\right)\right].
\end{aligned}$$

Hence, $[\beta_1^n, \beta_2^n] = -[\beta_2^n, \beta_1^n]$. □

Next, let us show that the Jacobi identity also holds.

Lemma 6.4. Suppose β_1, β_2 and β_3 are vector fields in M. Then

$$[\beta_1^n, [[\beta_2^n, \beta_3]] + [\beta_2^n, [\beta_3^n, \beta_1^n]] + [\beta_3^n, [\beta_1^n, \beta_2^n]] = 0.$$

Proof. Let φ_{1t} be the flow of β_1 in M, φ_{2t} the flow of β_2 in M and φ_{3t} the flow of β_3 in M, where the flows φ_{1t}, φ_{2t} and φ_{3t} are defined in $\mathcal{J}^n(M)$ respectively by the formulae

$$\beta_1^n = \varphi_{1t}\left([f]_{p_1}^n\right) = [\varphi_{1t}]_{f(p_1)}^n \cdot [f]_{p_1}^n = [\varphi_{1t} \circ f]_{p_1}^n$$

and

$$\beta_2^n = \varphi_{2t}\left([f]_{p_2}^n\right) = [\varphi_{2t}]_{f(p_2)}^n \cdot [f]_{p_2}^n = [\varphi_{2t} \circ f]_{p_2}^n$$

and

$$\beta_3^n = \varphi_{3t}\left([f]_{p_3}^n\right) = [\varphi_{3t}]_{f(p_3)}^n \cdot [f]_{p_3}^n = [\varphi_{3t} \circ f]_{p_3}^n$$

for all $p_1, p_2, p_3 \in M$. Then $[\beta_1, [\beta_2, \beta_3]]$ implies

$$\begin{aligned}
&\left[[\varphi_{1t} \circ f]_{p_1}^n, \left[[\varphi_{2t} \circ f]_{p_2}^n, [\varphi_{3t} \circ f]_{p_3}^n\right]\right] \\
&= \left[[\varphi_{1t} \circ f]_{p_1}^n, [\varphi_{2t} \circ f]_{p_2}^n [\varphi_{3t} \circ f]_{p_3}^n - [\varphi_{3t} \circ f]_{p_3}^n [\varphi_{2t} \circ f]_{p_2}^n\right] \\
&= [\varphi_{1t} \circ f]_{p_1}^n [\varphi_{2t} \circ f]_{p_2}^n [\varphi_{3t} \circ f]_{p_3}^n - [\varphi_{3t} \circ f]_{p_3}^n [\varphi_{2t} \circ f]_{p_2}^n [\varphi_{1t} \circ f]_{p_1}^n \\
&= [\varphi_{1t}]_{f(p_1)}^n \cdot [f]_{p_1}^n [\varphi_{2t}]_{f(p_2)}^n \cdot [f]_{p_2}^n [\varphi_{3t}]_{f(p_3)}^n \cdot [f]_{p_3}^n \\
&- [\varphi_{3t}]_{f(p_3)}^n \cdot [f]_{p_3}^n [\varphi_{2t}]_{f(p_2)}^n \cdot [f]_{p_2}^n [\varphi_{1t}]_{f(p_1)}^n \cdot [f]_{p_1}^n
\end{aligned}$$

At the same time, $[\beta_2,[\beta_3,\beta_1]]$ implies

$$\left[[\varphi_{2t}\circ f]^n_{p_2}, \left[[\varphi_{3t}\circ f]^n_{p_3}, [\varphi_{1t}\circ f]^n_{p_1}\right]\right]$$
$$= \left[[\varphi_{2t}\circ f]^n_{p_2}, [\varphi_{3t}\circ f]^n_{p_3}[\varphi_{1t}\circ f]^n_{p_1} - [\varphi_{1t}\circ f]^n_{p_1}[\varphi_{3t}\circ f]^n_{p_3}\right]$$
$$= [\varphi_{2t}\circ f]^n_{p_2}[\varphi_{3t}\circ f]^n_{p_3}[\varphi_{1t}\circ f]^n_{p_1} - [\varphi_{1t}\circ f]^n_{p_1}[\varphi_{3t}\circ f]^n_{p_3}[\varphi_{2t}\circ f]^n_{p_2}$$
$$= [\varphi_{2t}]^n_{f(p_2)}[f]^n_{p_2}[\varphi_{3t}]^n_{f(p_3)}[f]^n_{p_3}[\varphi_{1t}]^n_{f(p_1)}[f]^n_{p_1}$$
$$- [\varphi_{1t}]^n_{f(p_1)}[f]^n_{p_1}[\varphi_{3t}]^n_{f(p_3)}[f]^n_{p_3}[\varphi_{2t}]^n_{f(p_2)}[f]^n_{p_2}$$
$$= [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot\left([f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3}\right)$$
$$- [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot\left([f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3}\right)$$
$$= [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot\left([f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3} - [f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3}\right)$$
$$= [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot 0 = 0.$$

And lastly $[\beta_3^n,[\beta_1^n,\beta_2^n]]$ implies

$$\left[[\varphi_{3t}\circ f]^n_{p_3}, \left[[\varphi_{1t}\circ f]^n_{p_1}, [\varphi_{2t}\circ f]^n_{p_2}\right]\right]$$
$$= \left[[\varphi_{3t}\circ f]^n_{p_3}, [\varphi_{1t}\circ f]^n_{p_1}[\varphi_{2t}\circ f]^n_{p_2} - [\varphi_{2t}\circ f]^n_{p_2}[\varphi_{1t}\circ f]^n_{p_1}\right]$$
$$= [\varphi_{3t}\circ f]^n_{p_3}[\varphi_{1t}\circ f]^n_{p_1}[\varphi_{2t}\circ f]^n_{p_2} - [\varphi_{2t}\circ f]^n_{p_2}[\varphi_{1t}\circ f]^n_{p_1}[\varphi_{3t}\circ f]^n_{p_3}$$
$$= [\varphi_{3t}]^n_{f(p_3)}[f]^n_{p_3}[\varphi_{1t}]^n_{f(p_1)}[f]^n_{p_1}[\varphi_{2t}]^n_{f(p_2)}[f]^n_{p_2}$$
$$- [\varphi_{2t}]^n_{f(p_2)}[f]^n_{p_2}[\varphi_{1t}]^n_{f(p_1)}[f]^n_{p_1}[\varphi_{3t}]^n_{f(p_3)}[f]^n_{p_3}$$
$$= [\varphi_{3t}]^n_{f(p_3)}[\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}\cdot\left([f]^n_{p_3}[f]^n_{p_1}[f]^n_{p_2}\right)$$
$$- [\varphi_{2t}]^n_{f(p_2)}[\varphi_{1t}]^n_{f(p_1)}[\varphi_{3t}]^n_{f(p_3)}\left([f]^n_{p_2}[f]^n_{p_1}[f]^n_{p_3}\right)$$
$$= [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot\left([f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3}\right)$$
$$- [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot\left([f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3}\right)$$
$$= [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot\left([f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3} - [f]^n_{p_1}[f]^n_{p_2}[f]^n_{p_3}\right)$$
$$= [\varphi_{1t}]^n_{f(p_1)}[\varphi_{2t}]^n_{f(p_2)}[\varphi_{3t}]^n_{f(p_3)}\cdot 0 = 0.$$

Hence, $[\beta_1,[\beta_2,\beta_3]]+[\beta_2,[\beta_3,\beta_1]]+[\beta_3,[\beta_1,\beta_2]]=0.$ □
From the foregoing proposition 6.3 is established.

7. The Colom Beau Algebra

Distributions (see [4],[14], [10]) are presented by means of certain regularizations with model delta nets, that is they will be considered as parametrized families $(f*\phi)_{\phi\in\mathcal{I}_0}$ where $\mathcal{I}_0$ is a subspace of

$$\left\{\phi\in\mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n}\phi(t)dt = 1\right\}.$$

The main focus naturally, will be on the subfamilies

$$(f_\epsilon)_\epsilon = (f*\phi_\epsilon)_{\epsilon>0}$$

with ϕ_ϵ given by

$$\phi_\epsilon(t) = \epsilon^{-n}\phi\left(\frac{t}{\epsilon}\right).$$

To make the later differential-algebraic constructions work, we need to introduce an evaluation on the set of parameter.

Definition 7.1. Let $\phi\in\mathcal{D}(\mathbb{R}^n)$ be a test function, we define the parameter $\mathcal{I}=(0,1)$ as follows:

$$\mathcal{I}(\mathbb{R}^n) = \left\{\phi \in \mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \phi(t)dt = 1\right\} \tag{7.1}$$

and

$$\mathcal{I}(\mathbb{R}^n) = \left\{\phi \in \mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} t^\alpha \phi(t)dt = 0\right\} \tag{7.2}$$

for all $|\alpha| \geq 1$, where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbf{N}^n$, and $t^\alpha = (t_1)^{\alpha_1} \cdots (t_n)^{\alpha_n}$. This parameter is called a mollifier.

We have stated that for $\epsilon > 0$, $\phi_\epsilon(t) = \frac{1}{\epsilon^n}\phi(\frac{1}{\epsilon})$. Now, if $\phi_\epsilon(t) \geq 0$, then $\text{supp}(\phi_\epsilon(t)) = \overline{\mathbb{B}\,(0, \epsilon)}$, that is

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_\epsilon(t)dt &= \int_{\mathbb{R}^n} \frac{1}{\epsilon^n}\phi(\frac{t}{\epsilon})dt \\ &= \frac{1}{\epsilon^n}\int_{\mathbb{R}^n} \epsilon^n \phi(x)dx \\ &= \int_{\mathbb{R}^n} \phi(x)dx = 1 = \int_{\mathbb{R}^n} \phi(t)dt.\end{aligned}$$

Here we have used the change of variable $t = \epsilon x$ and $dt = \epsilon^n dx$. ϕ.

Definition 7.2. We define

$$\begin{aligned}\mathcal{E}\,(\mathbb{R}^n) &= \left(\mathcal{C}^\infty\,(\mathbb{R}^n)\right)^{\mathcal{I}(\mathbb{R}^n)} \\ &= \{f : \mathcal{I}\,(\mathbb{R}^n) \to \mathcal{C}^\infty\,(\mathbb{R}^n)\}\end{aligned}$$

as the set of all $\mathcal{C}^\infty-$functions in t for each fixed $\phi \in \mathcal{I}\,(\mathbb{R}^n)$.

Definition 7.3. The subset $\mathcal{E}_M\,(\mathbb{R}^n)$ of all $(f * \phi_\epsilon)_\epsilon = (f_\epsilon)_\epsilon \in \mathcal{E}\,(\mathbb{R}^n)$ such that: For all compact subsets K of $\mathbb{R}^n$, for all $\alpha \in \mathbf{N}_0^n$ there exists $N \in \mathbb{N}$ such that the seminorm

$$p_N\,(f) = \sup_{t\in K} |\partial^\alpha f_\epsilon(t)| = \mathcal{O}\left(\epsilon^{-N}\right) \leq c\epsilon^{-N} \tag{7.3}$$

as $\epsilon \to 0$ holds, where $c > 0$ and $\phi \in \mathcal{I}\,(\mathbb{R}^n)$. The elements of $\mathcal{E}_M\,(\mathbb{R}^n)$ are called moderate. They also constitute a differential algebra.

Definition 7.4. The set $\mathcal{N}\,(\mathbb{R}^n)$ of all $(f_\epsilon)_\epsilon \in \mathcal{E}\,(\mathbb{R}^n)$ with the property that: For all compact subset K of $\mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$ there exists $m \in \mathbb{N}$ such that the seminorm

$$p_m(f) = \sup_{t\in K} |\partial^\alpha f_\epsilon(t)| = \mathcal{O}\left(\epsilon^m\right) \leq c\epsilon^m \tag{7.4}$$

as $\epsilon \to 0$ holds, where $c > 0$ and $\phi \in \mathcal{I}\,(\mathbb{R}^n)$. The elements of $\mathcal{N}\,(\mathbb{R}^n)$ are called neutral function and tends to zero faster than any power of ϵ when evaluated at ϕ_ϵ with $\phi \in \mathcal{I}\,(\mathbb{R}^n)$ large enough. Clearly, $\mathcal{N}\,(\mathbb{R}^n)$ is a subalgebra closed under differentiation and it is an ideal of $\mathcal{E}_M\,(\mathbb{R}^n)$.

Definition 7.5. The algebra of generalized functions of Colombeau, denoted by $\mathcal{G}(\mathbb{R}^n)$ (or $\mathcal{G}$), is the quotient algebra

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_m(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n). \tag{7.5}$$

We remark that

- if $\overline{f}$ is a generalized function in $\mathcal{G}(\mathbb{R}^n)$ then

$$\begin{aligned}\overline{f} &= (f * \phi_\epsilon)_\epsilon + \mathcal{N}(\mathbb{R}^n) \\ &= (f_\epsilon)_\epsilon + \mathcal{N}\,(\mathbb{R}^n)\end{aligned}$$

, where $(f_\epsilon)_\epsilon \in \mathcal{E}_m(\mathbb{R}^n)$ is a representative of $\overline{f}$.

- Also, if $\overline{f} = \overline{g}$ in $\mathcal{G}(\mathbb{R}^n)$ then

$$(f * \phi_\epsilon)_\epsilon - (g * \phi_\epsilon)_\epsilon = (f_\epsilon)_\epsilon - (g_\epsilon)_\epsilon \in \mathcal{N}(\mathbb{R}^n)$$

where, $(f_\epsilon)_\epsilon$, $(g_\epsilon)_\epsilon$ are representatives of $\overline{f}$, $\overline{g}$ respectively.

Lemma 7.1. $\mathcal{G}(\mathbb{R}^n)$ is an associative and a commutative algebra with identity.

Proof. For all $f, \overline{g} \in \mathcal{G}(\mathbb{R}^n)$, we have

$$\begin{aligned}\overline{f}\overline{g}(\phi) &= ((f * \phi_\epsilon)_\epsilon + \mathcal{N}) \cdot ((g * \phi_\epsilon)_\epsilon + \mathcal{N}) \\ &= (f * \phi_\epsilon)_\epsilon \cdot (g * \phi_\epsilon)_\epsilon + \mathcal{N} \\ &= ((fg) * \phi_\epsilon)_\epsilon + \mathcal{N} \\ &= (f_\epsilon g_\epsilon)_\epsilon + \mathcal{N} \\ &= (f_\epsilon)_\epsilon (g_\epsilon)_\epsilon + \mathcal{N} \\ &\overline{fg}(\phi).\end{aligned}$$

It is obvious that $\partial^\alpha \mathcal{E}_m(\mathbb{R}^n) \subset \mathcal{E}_m(\mathbb{R}^n)$ and $\partial^\alpha \mathcal{N}(\mathbb{R}^n) \subset \mathcal{N}(\mathbb{R}^n)$, for all α. Therefore, we can define

$$\partial^\alpha : \mathcal{G}(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n) : \overline{f} \mapsto \partial^\alpha \overline{f} \tag{7.6}$$

where,

$$\begin{aligned}\partial^\alpha \overline{f} &= \partial^\alpha (f * \phi_\epsilon) + \mathcal{N}(\mathbb{R}^n) \\ &= \partial^\alpha f * \phi_\epsilon + \mathcal{N}(\mathbb{R}^n).\end{aligned}$$

It follows that ∂^α is linear, and satisfies Leibniz's rule of product derivatives. □

Lemma 7.2. $\mathcal{C}^0(\mathbb{R}^n)$ is included in $\mathcal{E}_M(\mathbb{R}^n)$ as a linear subspace, not a subalgebra. Consequently, $\mathcal{C}^0(\mathbb{R}^n)$ is not a subalgebra of $\mathcal{G}(\mathbb{R}^n)$, either.

Proof. In general, for all $f, g \in \mathcal{C}^0(\mathbb{R}^n)$ we have

$$\begin{aligned}(f * \phi_\epsilon)(g * \phi_\epsilon) &= \int_{\mathbb{R}^n} f(t)\phi_\epsilon(t-x)dt \int_{\mathbb{R}^n} g(t)\phi_\epsilon(t-x)dt \\ &= \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} f(t)\phi(\frac{t-x}{\epsilon})dt \int_{\mathbb{R}^n} g(t)\phi(\frac{t-x}{\epsilon})dt \\ &= \int_{\mathbb{R}^n} f(t+\epsilon y)\phi(y)dy \int_{\mathbb{R}^n} g(t+\epsilon y)\phi(y)dy \\ &\neq \int_{\mathbb{R}^n} f(t+\epsilon y)\, g(t+\epsilon y)\, \phi(y)dy \\ &= \int_{\mathbb{R}^n} fg(t+\epsilon y)\phi(y)dy \\ &= fg * \phi_\epsilon.\end{aligned}$$

Another issue that arise here is that, if $f \in \mathcal{C}^0(\mathbb{R}^n)$, we can show that $\tilde{f}_1 - \tilde{f}_2 \in \mathcal{N}(\mathbb{R}^n)$, so both $\tilde{f}_1$ and $\tilde{f}_2$ are representatives of $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. For convenience, we will show this in the case $n = 1$. Indeed, we have

$$\left(\tilde{f}_1 - \tilde{f}_2\right)\phi(t) = f(t) - \int_{\mathbb{R}} f(t+x)\,\phi(x)dx.$$

Therefore, one gets

$$\begin{aligned}\left(\tilde{f}_1 - \tilde{f}_2\right)(\phi_\epsilon) &= f(t) - \int_{\mathbb{R}} f(t+\epsilon y)\,\phi(y)\,dy \\ &= -\int_{\mathbb{R}} [f(t+\epsilon y) - f(t)]\,\phi(y)\,dy.\end{aligned}$$

Since $f \in \mathcal{C}^\infty(\mathbb{R})$, we can apply Taylor's formula up to order 1 to f at the point y, and we get

$$f(t+\epsilon y)-f(t)=(\epsilon y)\partial f"(t)+\frac{(\epsilon y)^{1+1}}{1+1}\partial f^{1+1}(t+\theta\epsilon y)$$
$$=(\epsilon y)f'(t)+\epsilon^2\frac{y^2}{2}\partial f^{(2)}(t+\theta\epsilon y)$$
$$=(\epsilon y)f'(t)+\epsilon^2\frac{y^2}{2}\partial f^{(2)}(t+\theta\epsilon y)$$

where $0<\theta<1$. Hence, for arbitrary compact K and $\phi\in\mathcal{I}$, we have

$$\left(\tilde{f}_1-\tilde{f}_2\right)\phi_\epsilon(t)=\mathcal{O}\left(\epsilon^2\right)$$

as $\epsilon\to 0$, uniformly on K and $m=1$.

Also, applying Taylor's formula up to order m to f at the point y, we obtain

$$f(t+\epsilon y)-f(t)=\sum_{k=1}^{m}\frac{(\epsilon y)}{k!}\partial f^{(k)}(t)+\epsilon^{(m+1)}\frac{y^{m+1}}{(k+1)!}\partial f^{(k+1)}(t+\theta\epsilon y)$$
$$=\sum_{k=1}^{m}\frac{(\epsilon^m y)}{k!}\partial f^{(k)}(t)+\epsilon^{m+1}\frac{y^{m+1}}{(k+1)!}\partial f^{(k+1)}(t+\theta\epsilon y)$$

where $0<\theta<1$. Therefore for arbitrary $K\subset\mathbb{R}$ and for all $m\in\mathbb{N}$, we have

$$\left(\tilde{f}_1-\tilde{f}_2\right)\phi_\epsilon(t)=\mathcal{O}\left(\epsilon^{m+1}\right)$$

as $\epsilon\to 0$ uniformly on K and $\phi\in\mathcal{I}$. This agrees with definition 7.4 for $|\alpha|\le m$. This fact also holds for the estimate $\partial^\alpha\left(\tilde{f}_1-\tilde{f}_2\right)\phi_\epsilon(t)$, so $\tilde{f}_1-\tilde{f}_2\in\mathcal{N}(\mathbb{R}^n)$. $\square$

7.1 $\mathcal{L}^1(\mathbb{R}^n)$ Embedded in the Colombeau Generalized Functions

$\mathcal{L}^1(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$ in the distribution theory and hence, $\mathcal{L}^1(\mathbb{R}^n)$ −functions are tempered distributions. So, to $f\in\mathcal{L}^1(\mathbb{R}^n)$, there corresponds an element in $\mathcal{G}(\mathbb{R}^n)$, denoted by $\tilde{f}+\mathcal{N}$ where $\tilde{f}\in\mathcal{E}_M(\mathbb{R}^n)$. Now,

can we express $\tilde{f}$ in term of f more clearly. In fact, we can do as follows:

$$\tilde{f}*\phi(t)=\langle f(x),\phi(x-t)\rangle$$
$$=\int_{\mathbb{R}^n}f(x)\phi(x-t)dx,$$

since $f\in\mathcal{L}^1(\mathbb{R}^n)$. So, we have $\tilde{f}+\mathcal{N}$ as the corresponding element of $f\in\mathcal{L}^1(\mathbb{R}^n)$. However, we cannot conclude from this that $\tilde{f}+\mathcal{N}_\tau\in\mathcal{G}_\tau(\mathbb{R}^n)$.

Since $f\in\mathcal{L}^1(\mathbb{R}^n)$, so the function g, where

$$g(t)=\int_{-\infty}^{t}f(x)dx,$$

$t\in\mathbb{R}^n$ is continuous on $\mathbb{R}^n$. To this end,

$$\|g(t)\|=\left|\int_{-\infty}^{t}f(y)dy\right|\le\int_{\mathbb{R}^n}|f(y)|\,dy=\|f\|_{\mathcal{L}^1}$$

for all $t\in\mathbb{R}^n$.

Hence, $g\in\mathcal{C}_\tau(\mathbb{R}^n)$. In $\mathcal{G}_\tau(\mathbb{R}^n)$, g is assigned with $\tilde{g}+\mathcal{N}_\tau(\mathbb{R}^n)$, where

$$\tilde{g}*\phi(t)=\int_{\mathbb{R}^n}g(t+x)\phi(x)dx.$$

It implies that f belongs to $\mathcal{G}_\tau(\mathbb{R}^n)$, and it is assigned with $\partial\tilde{g} + \mathcal{N}_\tau(\mathbb{R}^n)$. Therefore, f is assigned with the element

$$\partial_t\left(\int_{\mathbb{R}^n}\left(\int_{-\infty}^{t+x} f(y)dy\right)\phi(x)dx\right) + \mathcal{N}_\tau(\mathbb{R}^n).$$

Now, we notice that $\phi \in \mathcal{I}$ and $f \in \mathcal{L}^1(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t+x} f(y)dy\right)\phi(x)dx &= \int_{-\infty}^{\infty}\left(\int_{-\infty} tf(t+x)dy\right)\phi(x)dx\\ &= \int_{-\infty}^{t}\left(\int_{-\infty}^{\infty}\phi(x)f(y+x)dx\right)dy,\end{aligned}$$

and the inner integral as the function of y is in $\mathcal{L}^1(\mathbb{R}^n)\bigcap\mathcal{C}^\infty(\mathbb{R}^n)$. it follows that

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t+x} f(y)dy\right)\phi(x)dx = \int_{-\infty}^{\infty}\phi(x)f(t+x)dx.$$

Therefore, f is assigned with the element

$$\int_{\mathbb{R}^n} f(t+x)\phi(x)dx + \mathcal{N}_\tau$$

in $\mathcal{G}_\tau(\mathbb{R}^n)$. It also shows us that in $\mathcal{G}(\mathbb{R}^n)$, the function $f \in \mathcal{L}^1(\mathbb{R}^n)$ is assigned with the element

$$\int_{\mathbb{R}^n} f(t+x)\phi(x)dx + \mathcal{N}.$$

Now, we will use the results above to study the relationship between the integral of $f \in \mathcal{L}^1(\mathbb{R}^n)$ in the usual sense and the one in the sense of tempered generalized function.

Lemma 7.3. The topology of $\mathcal{G}(\mathbb{R}^n)$ is the topology it inherited as an embedding image of $\mathcal{L}^1(\mathbb{R}^n)$.

Proof. It suffices to show that for all $\overline{f} \in \mathcal{G}(\mathbb{R}^n)$, such that $\overline{f} = \tilde{f} + \mathcal{N}(\mathbb{R}^n)$, where $\tilde{f}$ is the representative of f in $\mathcal{E}_M(\mathbb{R}^n)$, for all $\phi \in \mathcal{I}(\mathbb{R}^n)$, and ϵ small enough we have

$$\left\|\tilde{f} * \phi_\epsilon(t)\right\|_{\mathcal{E}_M} \leq c\,\|f\|_{\mathcal{L}^1(\mathbb{R}^n)}.$$

To this end, we have

$$\begin{aligned}\left\|\tilde{f} * \phi_\epsilon(t)\right\|_{\mathcal{E}_M} &= \left|\int_{\mathbb{R}^n} f(t)\phi_\epsilon(t-x)dt\right|\\ &\leq \int_{\mathbb{R}^n}\left|f(t)\frac{1}{\epsilon^n}\phi\left(\frac{t-x}{\epsilon}\right)\right|dt\\ &\leq \epsilon^{-|n|}\int_{\mathbb{R}^n}|f(x+\epsilon y)\epsilon^n\phi(y)|\,dy\\ &\leq \epsilon^{|-n|}\cdot\epsilon^{|n|}\int_{\mathbb{R}^n}|f(x+\epsilon y)|\,|\phi(y)|\,dy\\ &\leq c\int_{\mathbb{R}^n}|f(x+\epsilon y)|\,dy\\ &\leq c\,\|f\|_{\mathcal{L}^1}.\end{aligned}$$

Therefore,

$$\left\|\tilde{f} * \phi_\epsilon(t)\right\|_{\mathcal{E}_M} = \mathcal{O}\left(\epsilon^0\right) \leq c\,\|f\|_{\mathcal{L}^1}$$

which means $\tilde{f} \in \mathcal{N}(\mathbb{R}^n)$ for all $n \in \mathbb{N}$.

Also, if

$$\widetilde{fg} + \mathcal{N} = \tilde{f}\tilde{g} + \mathcal{N} = \left(\tilde{f} + \mathcal{N}\right)(\tilde{g} + \mathcal{N})$$

where $\tilde{f}$ and $\tilde{g}$ are representatives of f and g in $\mathcal{E}_M(\mathbb{R}^n)$ we have

$$\begin{aligned}
\left\|\left(\widetilde{fg}\right) * \phi_\epsilon(t)\right\|_{\mathcal{E}_M} &= \left\|\left(\tilde{f}\tilde{g}\right) * \phi_\epsilon(t)\right\|_{\mathcal{E}_M} \\
&= \left\|\left(\tilde{f} * \phi_\epsilon(t)\right)(\tilde{g} * \phi_\epsilon(t))\right\|_{\mathcal{E}_M} \\
&\leq \left\|\tilde{f} * \phi_\epsilon(t)\right\|_{\mathcal{E}_M} \|\tilde{g} * \phi_\epsilon(t)\|_{\mathcal{E}_M} \\
&\leq c_1 \|f\|_{\mathcal{L}^1} \|\tilde{g} * \phi_\epsilon(t)\|_{\mathcal{E}_M} \\
&\leq c_1 c_2 \|f\|_{\mathcal{L}^1} \|g\|_{\mathcal{L}^1} \\
&\leq C \|f\|_{\mathcal{L}^1} \|g\|_{\mathcal{L}^1},
\end{aligned}$$

where we have taking $c_1 c_2 = C$, for all $\phi \in \mathcal{I}(\mathbb{R}^n)$, $0 < \epsilon < 1$ and $t \in \mathbb{R}^n$. $\square$

References

1. Blasco, O., Galindo, P., & Miralles, A. (2014). Bloch functions on the unit ball of an infinite dimensional Hilbert space. Journal of Functional Analysis, 267(4), 1188-1204.
2. Böttcher, A., & Silbermann, B. (2006). Analysis of Toeplitz operators. Berlin, Heidelberg: Springer Berlin Heidelberg.
3. Blecher, D. P., & Le Merdy, C. (2004). Operator algebras and their modules: an operator space approach (No. 30). Clarendon Press.
4. Colombeau, J. F. (1992). Multiplication of distributions: a tool in mathematics, numerical engineering, and theoretical physics. (No Title).
5. Egwe, M. E. (2020). On Fixed Point Theorem in Non-Archimedean Fuzzy Normed Spaces. International Journal of Analysis and Applications, 18(1), 99-103.
6. Egwe, M. E., & Oyewo, R. A. (2021). A fixed point theorem on fuzzy locally convex spaces. arXiv preprint arXiv:2101.12007.
7. Egwe, M.E. (2022). Fuzzy Locally Convex Spaces preprints.org.
8. Helgason, S. (2022). Groups and geometric analysis: integral geometry, invariant differential operators, and spherical functions (Vol. 83). American Mathematical Society.
9. Kaniuth, E. (2009). A course in commutative Banach algebras (Vol. 246). New York: Springer.
10. Tri, T. N. (2005). The Colombeau theory of generalized functions. University of Amsterdam.
11. Larsen., R. (1973). Banach Algebras. An Introduction. New York. Marcel Dekker, Inc.
12. Loomis., L.H. (1953). An Introduction to Abstract Harmonic Analysis. New Jersey, Princeton.
13. Murphy, G. J. (2014). C*-algebras and operator theory. Academic press.
14. Oberguggenberger, M. (1992). Multiplication of distributions and applications to partial differential equations. (No Title).
15. Pavlović, M. (2004). Introduction to function spaces on the disk. Belgrade.
16. Egwe, M. E., & Yusuf, F. (2023). A Note on Some Classes of Function Algebras. arXiv preprint arXiv:2401.06131.
17. Wolf, J. A. (2007). Harmonic analysis on commutative spaces (No. 142). American Mathematical Soc..
18. Xu, Z. (2019). Bloch type spaces on the unit ball of a Hilbert space. Czechoslovak Mathematical Journal, 69(3), 695-711.
19. Yumaguzhin, V. (2005). Introduction to differential invariants. Draft, Opava.
20. Zhu, K. (2005). Spaces of holomorphic functions in the unit ball. New York, NY: Springer New York.
21. Zhu, K. (2007). Operator theory in function spaces (No. 138). American Mathematical Soc.