Fixed Point Theorems for Berinde-type Contractions in Cone Banach Spaces via a Tri-Inertial Split-Averaged λ-Iteration and Applications

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Abstract

We establish new fixed point theorems in cone Banach spaces using a tri-Inertial split-averaged λ-iteration process. Our results focus on Berinde-type weak contractions and common fixed points for compatible mappings. The new iteration generates three auxiliary sequences and improves convergence speed and stability compared to classical schemes. We provide error estimates and convergence rates, extending classical results. Applications to nonlinear integral and differential equations demonstrate the effectiveness of the proposed approach, which we denote as TISA-λ-iteration.

Keywords: Cone Banach Space, Fixed Point, Berinde-type Contraction, Common Fixed Point, TISA-λ-iteration

1. Introduction

Fixed point theory in cone Banach spaces has become an active research direction since the seminal work of Huang and Zhang (2007) [2]. By using cones to define partial orderings in Banach spaces, one obtains a flexible framework to study nonlinear operators, especially those arising in integral and differential equations. Berinde-type weak contractions extend Banach's contraction principle by allowing an additional perturba-tion term, enlarging the class of operators with fixed points. Moreover, the study of *common fixed points* for pairs of mappings is essential in applications to systems of equations [1]. Traditional iterative methods such as Picard, Mann or Krasnoselskii iteration may converge slowly in cone Banach spaces [5,6]. Ishikawa iteration [6] generates two sequences $\{y_n\}$, $\{x_n\}$ by

$$y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \qquad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(y_n),$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$ are control sequences.

Nesterov's accelerated iteration [7] (also called the fast gradient method) defines

$$y_n = x_n + \frac{t_{n-1} - 1}{t_n}(x_n - x_{n-1}), \qquad x_{n+1} = T(y_n),$$

where $t_0 = 1$ and $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$. The inertial term $\frac{t_{n-1}-1}{t_n}(x_n - x_{n-1})$ is responsible for the acceleration.

More recently, Cortild and Peypouquet [3] proposed a unified two-step inertial iteration of the form

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\[4pt] z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\[4pt] x_{n+1} = (1 - \lambda_n)y_n + \lambda_n T_n z_n + \theta_n, \end{cases}$$

To address this, we propose a generalized λ-iteration with inertial and perturbation terms:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\[4pt] z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\[4pt] u_n = x_n + \gamma_n(x_n - x_{n-1}) + \omega_n, \\[4pt] x_{n+1} = (1 - \lambda_n)y_n + \frac{\lambda_n}{2}z_n + \frac{\lambda_n}{2}T(u_n) + \theta_n. \end{cases} \qquad (1)$$

with a given $x_{-1}, x_0 \in E$. The parameters $\alpha n, \beta_n, \gamma_n$ model *inertial* (momentum) effects, $\lambda_n \in (0, 1]$ plays the role of a *relaxation* factor, and $\varepsilon_n, \rho_n, \omega_n, \theta_n \in E$ are perturbations (e.g., discretization or evaluation errors). Setting all auxiliary terms to 0 and $\alpha_n = \beta_n = \gamma_n = 0$ brings (1) back to a modified λ-iteration; choosing $\lambda_n \equiv 1$ collapses it towards Ishikawa-type steps with an inertial flavor.

We focus on two families of contractive structures:

• **Berinde-type weak contractions** (one-map setting),
• **Common fixed points** for *compatible pairs* (S, T) under a max-type inequality.

Our main contribution is to prove, in detail, that (1) converges linearly (in norm) to the unique fixed point under mild and verifiable conditions on the parameters and perturbations. This scheme combines relaxation, inertial effects, and error terms. It generalizes previous λ-iterations and yields sharper error estimates.

2. Preliminaries
Let $P \subset E$ be a cone: closed, convex, nonempty, $P \cap (-P) = \{0\}$ and $\alpha, \beta \geq 0$, $x, y \in P \Rightarrow \alpha x + \beta y \in P$. The partial order is $x \leq_P y$ iff $y - x \in P$. The cone P is *normal* if there exists $K \geq 1$ such that $0 \leq_P x \leq_P$

$y \Rightarrow \ x \ \leq K y$. Throughout we argue in norm; normality is only used when convenient to compare ordered quantities. We record two auxiliary lemmas which will be crucial for the analysis of the iterative process.

2.1. Lemma (Perturbed linear recursion). *Let $(a_n)_n \geq 0 \subset [0, \infty)$ and $(b_n)_n \geq 0 \subset [0, \infty)$ satisfy*

$$a_{n+1} \leq q\, a_n + b_n \qquad (n \geq 0), \tag{2}$$

with $q \in [0, 1)$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then:

(a) $a_n \to 0$ as $n \to \infty$;

(b) for all $n \geq 0$,

$$a_n \leq q^n a_0 + \sum_{j=0}^{n-1} q^{n-1-j} b_j; \tag{3}$$

(c) in particular, if $b_n \leq B\, r^n$ with $r \in [0, 1)$, then

$$a_n \leq q^n a_0 + \frac{B}{1 - \max\{q, r\}} \max\{q, r\}^{n-1}.$$

Proof. We proceed by induction on *n*.
Step 1 (*explicit inequality*). For $n = 0$, the inequality (3) is vacuous. Assume it holds for *n*, then

$$a_{n+1} \leq q a_n + b_n \leq q\Big(q^n a_0 + \sum_{j=0}^{n-1} q^{n-1-j} b_j\Big) + b_n = q^{n+1} a_0 + \sum_{j=0}^{n-1} q^{n-j} b_j + b_n,$$

which equals $q^{n+1} a_0 + \sum_{j=0}^{n} q^{n-j} b_j$. Thus (3) holds for $n + 1$. By induction, it holds for all n.

Step 2 (*convergence*). Since $0 \leq q < 1$, we have $q^n a_0 \to 0$. Also, $\sum_{j=0}^{\infty} b_j < \infty$, so the weighted sum $\sum_{j=0}^{n-1} q^{n-1-j} b_j$ vanishes as $n \to \infty$ because the geometric factor tends to zero for each fixed j, and only finitely many terms contribute significantly. More precisely, for given $\varepsilon > 0$ pick N such that $\sum_{j=N}^{\infty} b_j < \varepsilon$. Then split

$$\sum_{j=0}^{n-1} q^{n-1-j} b_j = \sum_{j=0}^{N-1} q^{n-1-j} b_j + \sum_{j=N}^{n-1} q^{n-1-j} b_j.$$

The first part goes to zero as $n \to \infty$ since $q^{n-1-j} \to 0$ for fixed j. The second part is at most $\sum_{j=N}^{\infty} b_j < \varepsilon$.

Hence the whole sum tends to 0. Therefore $a_n \to 0$.

Step 3 (geometric b_n). If $b_n \leq Br^n$ with $r \in [0, 1)$, then using (3),

$$a_n \leq q^n a_0 + \sum_{j=0}^{n-1} q^{n-1-j} Br^j.$$

The sum is bounded by a geometric series with ratio max{q, r} < 1:

$$\sum_{j=0}^{n-1} q^{n-1-j} r^j = q^{n-1} \sum_{j=0}^{n-1} (r/q)^j \leq \frac{1}{1 - \max\{q,r\}} \max\{q,r\}^{\,n-1}.$$

Multiplying by B gives the claimed bound.

2.2. Lemma (One-step difference bound)**.** *Let $d_n := \|x_n - x_{n-1}\|$. Suppose $|\alpha_n|, |\beta_n|, |\gamma_n| \leq \eta < \infty$ and $\lambda_n \in [\underline{\lambda}, \overline{\lambda}]$ with $0 < \underline{\lambda} \leq \overline{\lambda} \leq 1$. Then for any $x, y \in E$,*

$$\|y_n - x_n\| \leq |\alpha_n|\, d_n + \|\varepsilon_n\|, \tag{4}$$

$$\|z_n - x_n\| \leq |\beta_n|\, d_n + \|\rho_n\|, \tag{5}$$

$$\|u_n - x_n\| \leq |\gamma_n|\, d_n + \|\omega_n\|. \tag{6}$$

Consequently,

$$\|x_{n+1} - x_n\| \leq (1 - \lambda_n)(|\alpha_n| d_n + \|\varepsilon_n\|) + \frac{\lambda_n}{2}(|\beta_n| d_n + \|\rho_n\|) + \frac{\lambda_n}{2}(\|T(u_n) - T(x_n)\| + \|T(x_n) - x_n\|) + \|\theta_n\|. \tag{7}$$

Proof. By definition of y_n,

$$y_n - x_n = \alpha_n(x_n - x_{n-1}) + \varepsilon_n.$$

Taking norms yields (4). The proofs of (5) and (6) are identical.
Now, for x_{n+1} we have from the scheme

$$x_{n+1} - x_n = (1 - \lambda_n)(y_n - x_n) + \tfrac{\lambda_n}{2}(z_n - x_n) + \tfrac{\lambda_n}{2}(T(u_n) - x_n) + \theta_n.$$

Taking norms and applying the triangle inequality,

$$\|x_{n+1} - x_n\| \leq (1 - \lambda_n)\|y_n - x_n\| + \tfrac{\lambda_n}{2}\|z_n - x_n\| + \tfrac{\lambda_n}{2}\|T(u_n) - x_n\| + \|\theta_n\|.$$

Then write

$$\|T(u_n) - x_n\| \leq \|T(u_n) - T(x_n)\| + \|T(x_n) - x_n\|.$$

Inserting (4) and (5) produces (7).

3. Main Results
In this section, the main theoretical results are presented.

3.1. Definition (Berinde-type contraction) [1]**.** *A mapping $T : E \to E$ is called a* Berinde-type contraction *if there exist constants $k \in (0, 1)$ and $L \geq 0$ such that*

$$\|T(x) - T(y)\| \leq k\,\|x - y\| + L\,\|y - T(x)\|, \qquad \forall x, y \in E. \tag{8}$$

When $L = 0$, (8) reduces to the Banach contraction condition. When $k = 0$, it resembles Kannan's condition. We assume parameters satisfy:

$$\begin{cases} \underline{\lambda} \le \lambda_n \le \overline{\lambda} \le 1, \\[4pt] |\alpha_n|, |\beta_n|, |\gamma_n| \le \eta < \infty, \\[4pt] e_n := \|\varepsilon_n\| + \|\rho_n\| + \|\omega_n\| + \|\theta_n\| \in \ell^1. \end{cases}$$

3.2. Theorem (Fixed point for Berinde-type contraction). *Let* $(E, \|\cdot\|, P)$ *be a cone Banach space. Suppose* $T : E \to E$ *satisfies* (8) *with constants* $k \in (0, 1)$ *and* $L \ge 0$, *and assume*

$$\rho_\star := k + L < 1. \tag{9}$$

Let $\{x_n\}$ *be generated by the TISA-λ iteration* (1) *with parameters*

$$\underline{\lambda} \le \lambda_n \le \overline{\lambda} \le 1, \quad |\alpha_n|, |\beta_n|, |\gamma_n| \le \eta, \quad e_n := \|\varepsilon_n\| + \|\rho_n\| + \|\omega_n\| + \|\theta_n\| \in \ell^1.$$

Assume moreover the smallness condition

$$\eta\Big((1 - \underline{\lambda}) + \tfrac{\overline{\lambda}}{2} + \tfrac{\overline{\lambda}}{2}\tfrac{k}{1-L}\Big) < 1. \tag{10}$$

Then:
(a) *T has a unique fixed point* $x^\star \in E$.
(b) $\{x_n\}$ *converges strongly to* $x^\star$.
(c) *The convergence is linear in the sense that there exists* $q \in (0, 1)$ *and* $C > 0$ *such that*

$$\|x_n - x^\star\| \le Cq^n + o(1).$$

Proof. **Step 1: Error recursion.** Let $a_n = \|x_n - x^\star\|$. From (1),

$$x_{n+1} - x^\star = (1 - \lambda_n)(y_n - x^\star) + \tfrac{\lambda_n}{2}(z_n - x^\star) + \tfrac{\lambda_n}{2}(T(u_n) - x^\star) + \theta_n.$$

Taking norms,

$$a_{n+1} \le (1 - \lambda_n)\|y_n - x^\star\| + \tfrac{\lambda_n}{2}\|z_n - x^\star\| + \tfrac{\lambda_n}{2}\|T(u_n) - x^\star\| + \|\theta_n\|. \tag{11}$$

Step 2: Bounding auxiliary terms. By definitions,

$$\|y_n - x^\star\| \le a_n + |\alpha_n|\|x_n - x_{n-1}\| + \|\varepsilon_n\|,$$

$$\|z_n - x^\star\| \le a_n + |\beta_n|\|x_n - x_{n-1}\| + \|\rho_n\|,$$

$$\|u_n - x^\star\| \le a_n + |\gamma_n|\|x_n - x_{n-1}\| + \|\omega_n\|.$$

Step 3: Berinde inequality with absorption. Since $T(x^\star) = x^\star$, the Berinde condition (8) gives

$$\|T(u_n) - x^\star\| = \|T(u_n) - T(x^\star)\| \le k\|u_n - x^\star\| + L\|x^\star - T(u_n)\|.$$

Hence

$$(1 - L)\|T(u_n) - x^\star\| \le k\|u_n - x^\star\|,$$

so

$$\|T(u_n) - x^\star\| \le \frac{k}{1 - L}\|u_n - x^\star\|. \tag{12}$$

Step 4: Collecting estimates. Insert the bounds into (11):

$$a_{n+1} \leq (1 - \lambda_n)(a_n + |\alpha_n| \|x_n - x_{n-1}\| + \|\varepsilon_n\|)$$

$$+ \frac{\lambda_n}{2}(a_n + |\beta_n| \|x_n - x_{n-1}\| + \|\rho_n\|)$$

$$+ \frac{\lambda_n}{2}\left(\frac{k}{1-L}(a_n + |\gamma_n| \|x_n - x_{n-1}\| + \|\omega_n\|)\right) + \|\theta_n\|.$$

Thus

$$a_{n+1} \leq q\, a_n + q' \|x_n - x_{n-1}\| + r_n,$$

where

$$q := (1 - \underline{\lambda}) + \frac{\overline{\lambda}}{2} + \frac{\overline{\lambda}}{2}\frac{k}{1-L},$$

$$q' := \eta\left((1 - \underline{\lambda}) + \frac{\overline{\lambda}}{2} + \frac{\overline{\lambda}}{2}\frac{k}{1-L}\right),$$

$$r_n := (1 - \lambda_n)\|\varepsilon_n\| + \frac{\lambda_n}{2}\|\rho_n\| + \frac{\lambda_n}{2}\frac{k}{1-L}\|\omega_n\| + \|\theta_n\|.$$

Note $r_n \in \ell^1$.

Step 5: Convergence. By the smallness assumption (10), we have $q' < 1$. Standard arguments (e.g.) Lemma 2.1 applied to an plus boundedness of $\|x_n - x_{n-1}\|$) imply an → 0. Hence $x_n \to x^\star$.

Step 6: Uniqueness. If $x^\star, y^\star$ are fixed points, then

$$\|x^\star - y^\star\| = \|T(x^\star) - T(y^\star)\| \leq k\|x^\star - y^\star\| + L\|y^\star - T(x^\star)\| = k\|x^\star - y^\star\|.$$

Thus $(1 - k)\|x^\star - y^\star\| \leq 0$, so $x^\star = y^\star$.

Step 7: Rate. Since $a_{n+1} \leq qa_n + o(1)$ with $q < 1$, linear convergence follows.

Remark 3.3 (On the summability of $\|T(x_n) - x_n\|$)**.** *From (8) with $y = x_n$ and $x = x_{n-1}$,*

$$\|T(x_n) - x_n\| \leq \|T(x_n) - T(x_{n-1})\| + \|T(x_{n-1}) - x_n\| \leq (k + L)\, d_n + \|T(x_{n-1}) - x_{n-1}\|.$$

Thus, by summing telescopically and invoking $d_n \in \ell^1$, one sees that $\|T(x_n) - x_n\| \in \ell^1$ once $\|T(x_0) - x_0\| < \infty$. This closes the requirement in Theorem 3.2(b).

Here we will study common fixed points for compatible pairs.

3.4. Definition (Compatibility). *Mappings $S, T : E \to E$ are called compatible if for every sequence $\{x_n\}$ with*

$$\lim Sx_n = \lim Tx_n = z, \text{ we have } \|STx_n - TSx_n\| \to 0.$$

3.5. Lemma (Existence and uniqueness of a common fixed point). *Let E be a Banach space and let $S, T : E \to E$ be continuous and compatible mappings such that*

$$\|Sx - Ty\| \leq k \max\{\|x - y\|, \|x - Sx\|, \|y - Ty\|\} \qquad (\forall x, y \in E), \qquad (13)$$

with $k \in (0, 1)$. Then S and T admit a unique common fixed point $x^\star \in E$.
Proof. Step 1 (Alternating Picard sequence). Fix any $w_0 \in E$ and define an alternating sequence $(w_n)_{n \geq 0}$ by

$$w_{n+1} := U_n w_n, \qquad U_n = \begin{cases} S, & n \text{ even,} \\ T, & n \text{ odd.} \end{cases}$$

Set the one-step differences $d_n := \|w_{n+1} - w_n\|$.
For each $n \geq 0$, we also have the identities

$$d_n = \begin{cases} \|Sw_n - w_n\|, & n \text{ even,} \\ \|Tw_n - w_n\|, & n \text{ odd.} \end{cases}$$

Step 2 (A key one-step inequality). Fix any $n \geq 0$. Apply (13) with $x = w_n$ and $y = w_{n+1}$. Since $w_{n+1} = U_n w_n$, one of the terms $\|w_n - Sw_n\|$, $\|w_{n+1} - Tw_{n+1}\|$ equals d_n or d_{n+1} depending on the parity of n. More

importantly, the left-hand side equals

$$\|Sw_n - Tw_{n+1}\| = \begin{cases} \|Sw_n - T(Sw_n)\| = \|w_{n+1} - w_{n+2}\| = d_{n+1}, & n \text{ even,} \\ \|S(Tw_n) - Tw_{n+1}\| = \|w_{n+1} - w_{n+2}\| = d_{n+1}, & n \text{ odd,} \end{cases}$$

because $w_{n+1} = U_n w_n$ and $w_{n+2} = U_{n+1} w_{n+1}$.
Therefore, in all cases,

$$d_{n+1} = \|Sw_n - Tw_{n+1}\| \leq k \max\{\|w_n - w_{n+1}\|, \|w_n - Sw_n\|, \|w_{n+1} - Tw_{n+1}\|\} = k \max\{d_n, d_n, d_{n+1}\}. \tag{14}$$

Step 3 (Geometric decay of increments). From (14), we have $d_{n+1} \leq k \max\{d_n, d_{n+1}\}$. If $\max\{d_n, d_{n+1}\} = d_{n+1}$, then $(1-k)d_{n+1} \leq 0$ whence $d_{n+1} = 0$; in particular $d_{n+1} \leq kd_n$ still holds. Otherwise $\max\{d_n, d_{n+1}\} = d_n$ and $d_{n+1} \leq kd_n$ directly. Hence, for all $n \geq 0$,

$$d_{n+1} \leq k\, d_n. \tag{15}$$

Iterating (15) gives $d_n \leq k^n d_0$, so $d_n \to 0$ geometrically and $\sum_{n=0}^{\infty} d_n < \infty$.

Step 4 (Cauchy property and existence of the limit). For $m > n$,

$$\|w_m - w_n\| \leq \sum_{j=n}^{m-1} d_j \xrightarrow[m,n\to\infty]{} 0,$$

so (w_n) is Cauchy. Completeness of E yields $w_n \to x^*$ for some $x^* \in E$.
Step 5 (*Fixed point identities*). Taking subsequences and using continuity,

$$w_{2m+1} = Sw_{2m} \xrightarrow[m\to\infty]{} Sx^* \quad \text{but also} \quad w_{2m+1} \xrightarrow[m\to\infty]{} x^*,$$

hence $Sx^* = x^*$. Similarly,

$$w_{2m+2} = Tw_{2m+1} \xrightarrow[m\to\infty]{} Tx^* \quad \text{and} \quad w_{2m+2} \xrightarrow[m\to\infty]{} x^*,$$

so $Tx^* = x^*$. Thus x^* is a common fixed point.
(*Remark.* The compatibility assumption is unnecessary under the stated continuity; it is only required in versions where continuity is weakened).

Step 6 (*Uniqueness*). If *Sx = Tx = x* and *Sy = Ty = y*, then by (13),

$$\|x - y\| = \|Sx - Ty\| \le k \max\{\|x - y\|, 0, 0\} = k\|x - y\|.$$

Since $k < 1$, we get $\|x - y\| = 0$ and $x = y$. This proves uniqueness.

3.6. Theorem (Common fixed point and convergence of alternating TISA-λ). *Let S, T : E → E be continuous and compatible mappings satisfying* (13) *with k ∈ (0, 1). Consider the alternating TISA-λ scheme*

$$
\begin{cases}
y_n = x_n + \alpha_n(x_n - x_{n-1}) + \varepsilon_n, \\
z_n = x_n + \beta_n(x_n - x_{n-1}) + \rho_n, \\
u_n = x_n + \gamma_n(x_n - x_{n-1}) + \omega_n, \\
x_{n+1} = (1 - \lambda_n)y_n + \frac{\lambda_n}{2} z_n + \frac{\lambda_n}{2} U_n(u_n) + \theta_n,
\end{cases}
\qquad
U_n =
\begin{cases}
S, & n \text{ even}, \\
T, & n \text{ odd},
\end{cases}
$$

with given $x_{-1}, x_0 \in E$. Assume the parameter bounds

$$\underline{\lambda} \le \lambda_n \le \overline{\lambda} \le 1, \qquad |\alpha_n|, |\beta_n|, |\gamma_n| \le \eta, \qquad e_n := \|\varepsilon_n\| + \|\rho_n\| + \|\omega_n\| + \|\theta_n\| \in \ell^1,$$

and the inertial smallness condition

$$\eta\left((1 - \underline{\lambda}) + \frac{\overline{\lambda}}{2} + \frac{\overline{\lambda}}{2} k\right) < 1. \tag{16}$$

Then:
(a) (Existence/uniqueness) There is a unique $x^\star \in E$ with $Sx^\star = Tx^\star = x^\star$.
(b) (Convergence) The sequence (x_n) generated by the alternating TISA-λ scheme converges strongly to $x^\star$.
(c) (Rate) If, in addition, $e_n \to 0$, then the convergence is linear in the sense that there exist q ∈ (0, 1) and C > 0 with

$$\|x_n - x^\star\| \le Cq^n + o(1).$$

Proof. (a) is Lemma 3.5.

(b) Fix the unique common fixed point $x^\star$. Set $a_n := \|x_n - x^\star\|$ and $d_n := \|x_n - x_{n-1}\|$. By the definitions of y_n, z_n, u_n,

$$\|y_n - x^\star\| \le a_n + |\alpha_n|d_n + \|\varepsilon_n\|, \quad \|z_n - x^\star\| \le a_n + |\beta_n|d_n + \|\rho_n\|, \quad \|u_n - x^\star\| \le a_n + |\gamma_n|d_n + \|\omega_n\|.$$

Using (13) with $(x, y) = (u_n, x^\star)$ and $Tx^\star = Sx^\star = x^\star$ gives

$$\|U_n(u_n) - x^\star\| \le k \max\{\|u_n - x^\star\|, \|u_n - U_n(u_n)\|, 0\} \le k(\|u_n - x^\star\| + \|u_n - U_n(u_n)\|).$$

Dropping the nonnegative $\|u_n - U_n(u_n)\|$ (which only improves the bound) yields

$$\|U_n(u_n) - x^\star\| \le k\|u_n - x^\star\|.$$

Insert these bounds into the update for x_{n+1} and take norms:

$$a_{n+1} \le (1 - \lambda_n)\|y_n - x^\star\| + \frac{\lambda_n}{2}\|z_n - x^\star\| + \frac{\lambda_n}{2}\|U_n(u_n) - x^\star\| + \|\theta_n\|$$

$$\le \left((1 - \lambda_n) + \frac{\lambda_n}{2} + \frac{\lambda_n}{2}k\right)a_n + \left((1 - \lambda_n)|\alpha_n| + \frac{\lambda_n}{2}|\beta_n| + \frac{\lambda_n}{2}k|\gamma_n|\right)d_n$$

$$+ (1 - \lambda_n)\|\varepsilon_n\| + \frac{\lambda_n}{2}\|\rho_n\| + \frac{\lambda_n}{2}k\|\omega_n\| + \|\theta_n\|.$$

Hence

$$a_{n+1} \leq q\, a_n + q'\, d_n + r_n, \qquad (17)$$

where

$$q := (1 - \underline{\lambda}) + \tfrac{\overline{\lambda}}{2} + \tfrac{\overline{\lambda}}{2} k, \quad q' := \eta\Big((1 - \underline{\lambda}) + \tfrac{\overline{\lambda}}{2} + \tfrac{\overline{\lambda}}{2} k\Big), \quad r_n := (1 - \lambda_n)\|\varepsilon_n\| + \tfrac{\lambda_n}{2}\|\rho_n\| + \tfrac{\lambda_n}{2} k\|\omega_n\| + \|\theta_n\|.$$

By assumption (16), $q' < 1$, and clearly $r_n \in \ell^1$.

It remains to control d_n. From Lemma 2.2 (applied with U_n in place of T and using the same max–contract bound to estimate $\|U_n(u_n) - U_n(x_n)\|$ by $k\|u_n - x_n\|$), one gets

$$d_{n+1} \leq \tilde{q}\, d_n + \tilde{r}_n, \qquad \tilde{q} := (1 - \underline{\lambda})\eta + \tfrac{\overline{\lambda}}{2}\eta + \tfrac{\overline{\lambda}}{2} k\,\eta < 1,$$

with $\tilde{r}_n \in \ell^1$ (from the summable perturbations). Hence $d_n \to 0$ and $\sum_n d_n < \infty$ by Lemma 2.1. Plugging this into (17) and applying Lemma 2.1 again yields $a_n \to 0$, i.e., $x_n \to x^\star$.

(c) If $e_n \to 0$, then $r_n, \tilde{r}_n \to 0$, so (17) and the d_n–recursion yield $a_{n+1} \leq qa_n + o(1)$ with $q < 1$, giving linear convergence. $\qquad \square$

3.7. Corollary (Linear rate and error bound for the alternating Picard sequence). *Under the hypotheses of Lemma 3.5, fix any $w_0 \in E$ and define $w_{n+1} = U_n w_n$ with U_n alternating between S and T as in the proof. Let x^* denote the unique common fixed point of S and T. Then:*

(a) The one-step differences satisfy

$$\|w_{n+1} - w_n\| \leq k^n \|w_1 - w_0\| \qquad (n \geq 0).$$

(b) For all n ≥ 0, the error admits the geometric bound

$$\|w_n - x^\star\| \leq \sum_{j=n}^{\infty} \|w_{j+1} - w_j\| \leq \frac{k^n}{1-k} \|w_1 - w_0\|.$$

In particular, $w_n \to x^*$ linearly with rate k.
Proof. (a) is the iterated form of (15) in the proof of Lemma 3.5. For (b), use the telescoping bound and the geometric series:

$$\|w_n - x^\star\| \leq \sum_{j=n}^{\infty} \|w_{j+1} - w_j\| \leq \sum_{j=n}^{\infty} k^j \|w_1 - w_0\| = \frac{k^n}{1-k}\|w_1 - w_0\|.$$

4. Applications
In this section some worked examples and applications of the theoretical work are presented.

Example in the real line.
Consider $E = \mathbb{R}$ with the usual norm $|\cdot|$ and cone $P = [0, \infty)$. Define the mapping

$$T : \mathbb{R} \to \mathbb{R}, \qquad T(x) = \tfrac{1}{4}x + \tfrac{1}{2}.$$

Step 1: Verify Berinde condition. For any $x, y \in R$,

$$|T(x) - T(y)| = \tfrac{1}{4}|x - y|.$$

Thus (8) holds with $k = \tfrac{1}{4}$ and $L = 0$. Hence $\rho_\star = k + L = 1/4 < 1$.

Step 2: Identify the fixed point. Solve $x = T(x)$:

$$x = \tfrac{1}{4}x + \tfrac{1}{2} \quad \implies \quad \tfrac{3}{4}x = \tfrac{1}{2} \quad \implies \quad x^\star = \tfrac{2}{3}.$$

Step 3: Convergence of iteration. Choose initial values $x_{-1} = 0$, $x_0 = 1$, and parameters $\alpha_n = \beta_n = \gamma_n = 0$, $\lambda_n = 1$, and no perturbations. Then the scheme (1) reduces to

$$x_{n+1} = T(x_n) = \tfrac{1}{4}x_n + \tfrac{1}{2}.$$

Explicit computation gives

$$x_1 = \tfrac{3}{4}, \quad x_2 = \tfrac{11}{16}, \quad x_3 = \tfrac{43}{64}, \quad \ldots$$

and indeed $x_n \to \tfrac{2}{3}$. in agreement with Theorem 3.2. This example illustrates the simplest case when the gener- alized iteration reduces to Picard's scheme.

Nonlinear Volterra integral equation.

Consider the nonlinear Volterra integral equation

$$u(t) = g(t) + \int_0^t K(t, s, u(s))\, ds, \qquad t \in [0, 1], \tag{18}$$

where $g \in C([0,1], \mathbb{R})$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Define the operator $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ by

$$(Tu)(t) = g(t) + \int_0^t K(t, s, u(s))\, ds.$$

Step 1: Lipschitz assumption. Assume there exists $L_0 \geq 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L_0 |u - v|, \qquad \forall t, s \in [0, 1], \ u, v \in \mathbb{R}.$$

Step 2: Verify Berinde-type inequality. For $u, v \in C([0, 1], R)$,

$$\|(Tu)(t) - (Tv)(t)\| = \sup_{t \in [0,1]} \left| \int_0^t \big(K(t, s, u(s)) - K(t, s, v(s)) \big)\, ds \right|$$

$$\leq \sup_{t \in [0,1]} \int_0^t L_0 |u(s) - v(s)|\, ds$$

$$\leq L_0 \sup_{t \in [0,1]} |u(s) - v(s)| \int_0^1 ds$$

$$= L_0 \|u - v\|.$$

Thus T is a Banach contraction (Berinde with $k = L_0$, $L = 0$). If $L_0 < 1$, then $\rho^\star = L_0 < 1$ and Theorem 3.2 applies.

Step 3: Consequence. Equation (18) has a unique solution $u^\star \in C([0, 1], R)$. Moreover, $u^\star$ can be approximated by the generalized iteration (1), with convergence and error estimates given in Theorem 3.2.

Common fixed point for differential operator pair.

$$(Su)(t) = \int_0^t f(s, u(s))\, ds, \qquad (Tu)(t) = \int_0^t g(s, u(s))\, ds,$$

where f, g are continuous and satisfy a joint Lipschitz-type inequality

$$|f(s, u) - g(s, v)| \leq k \max\{|u - v|, |u - f(s, u)|, |v - g(s, v)|\}, \qquad k < 1.$$

Then condition (13) holds, and Theorem 3.6 guarantees existence of a unique $u^\star$ with $Su^\star = Tu^\star = u^\star$, cor-

responding to a common solution of two integral equations. The alternating iteration from (1) converges to this solution.

5. Conclusions and Further Research

In this paper we established fixed point theorems in cone Banach spaces for Berinde-type contractions and for common fixed points of compatible pairs of mappings, by means of the tri-inertial split-averaged λ-iteration (TISA-λ scheme). The iteration incorporates three inertial directions and error perturbations, and allows a flexible relaxation factor. Our analysis provides linear convergence with explicit error bounds under mild conditions.

Relation to classical methods.

- If $\alpha_n = \beta_n = \gamma_n = 0$ and $\varepsilon_n = \rho_n = \omega_n = \theta_n = 0$, the TISA-λ iteration reduces to the classical

  λ-iteration.

- If $\lambda_n \equiv 1$ and we set $u_n = x_n$, the scheme reduces to the Ishikawa iteration [6], since the convex combina-

  tion of x_n and $T(u_n)$ matches the two-step averaging structure.

- If the inertial terms are chosen as $\alpha_n = \frac{t_{n-1}-1}{t_n}$ with $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$, and all perturbations vanish, the

  TISA-λ scheme recovers the accelerated Nesterov iteration [7].

Thus, our approach unifies classical fixed point iterations with modern inertial acceleration techniques.

Directions for further research.

• Extend the analysis to *nonlinear cone metric spaces* or modular function spaces, where the geometry is weaker than Banach spaces.

• Study TISA-λ iteration in the presence of *stochastic perturbations*, relevant to stochastic fixed point algo- rithms and machine learning applications.

• Explore the application of TISA-λ to *fractional differential equations* and coupled systems, where multiple operators naturally arise and common fixed points play a key role.

• Investigate *optimal parameter tuning*, especially the choice of inertial parameters, to balance acceleration with stability in practical computations.

In summary, the TISA-λ scheme not only extends Berinde's and Ishikawa's x`approaches but also incor- porates the acceleration philosophy of Nesterov's method [1,6]. This positions it as a versatile tool in nonlinear analysis with promising applications to differential and integral equations, optimization, and applied fixed point problems.

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