

Global Dynamics of a Planar Polynomial Mechanical System

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Abstract

In this paper, using a bifurcational geometric approach, we study the global dynamics and solve the problem on the maximum number and distribution of limit cycles in a planar polynomial Euler–Lagrange–Liénard type mechanical system.

Keywords: *Euler–Lagrange–Liénard equation, mechanical system, planar polynomial dynamical system, bifurcation, field rotation parameter, singular point, limit cycle.*

I. Introduction

In this paper, we study an Euler–Lagrange–Liénard type equation

$$\ddot{x} + h(x) \dot{x}^2 + f(x) \dot{x} + g(x) = 0 \quad (1)$$

and the corresponding dynamical system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y - h(x) \dot{x}^2. \quad (2)$$

Equation (1) is a composition of two equations. One of them is

$$\alpha(q) \ddot{q} + \beta(q) \dot{q}^2 + \gamma(q) = 0, \quad (3)$$

where $q \in R$; $\alpha(q)$, $\beta(q)$ and $\gamma(q)$ are scalar functions, which represents a generic form of dynamics for an n -degree of freedom Euler–Lagrange system

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{Q}} \right) - \frac{\partial L}{\partial Q} = B(Q) u, \quad (4)$$

where $L(Q, \dot{Q})$ is a Lagrangian, $Q \in R^n$ is a vector of generalized coordinates, $u \in R^{n-1}$ and $B(Q)$ is $n \times (n-1)$ matrix function of full rank for each Q . Equation (3) can be used, in particular, for solving the periodic motion problem in mechanical systems; see, e. g., [26,27] and the references therein.

The other one is the Liénard equation (5)

$$\ddot{x} + f(x) \dot{x} + g(x) = 0 \quad (5)$$

with the corresponding dynamical systems in the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \quad (6)$$

particular cases of which we have considered in [9-16]; See also [6,7,17,19,20,22,28]. There are many examples in the natural sciences and technology in which this and related systems are applied [1,2,21,25]. Such systems are often used to model either mechanical or electrical, or

biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e.g., in certain mechanical systems, where $f(x)$ represents a coefficient of the damping force and $g(x)$ represents the restoring force or stiffness, when modeling wind rock phenomena and surge in jet engines [1,21]. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. The Liénard system has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [25].

There are also a number of examples of technical systems which are modelled with quadratic damping: a term in the second-order dynamics model, which is quadratic with respect to the velocity state variable. These examples include bearings, floating off-shore structures, vibration isolation and ship roll damping models [5,18]. In robotics, quadratic damping appears in feed-forward control and in nonlinear impedance devices, such as variable impedance actuators [3,4]. Variable impedance actuators are of particular interest for collaborative robotics [23,24].

We suppose that system (2), where $g(x)$, $h(x)$ and $f(x)$ are arbitrary polynomials, has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + a_1 x + \dots + a_{2l} x^{2l}) \\ &\quad + y(\alpha_0 + \alpha_1 x + \dots + \alpha_{2k} x^{2k}) \\ &\quad + y^2(c_0 + c_1 x + \dots + c_{2n} x^{2n}). \end{aligned} \quad (7)$$

Note that for $g(x) \equiv x$ and $h(x) \equiv 0$, by the change of variables $X = x$ and $Y = y + F(x)$, where $F(x) = \int_0^x f(s) ds$,

(7) is reduced to an equivalent system

$$\dot{X} = Y - F(X), \quad \dot{Y} = -X \quad (8)$$

which can be written in the form

$$\dot{x} = y, \quad \dot{y} = -x + F(y) \quad (9)$$

or

$$\begin{aligned} \dot{x} &= y, \\ &= -x + \gamma_1 y + \gamma_2 y^2 + \gamma_3 y^3 + \dots \\ &\quad + \gamma_{2k} y^{2k} + \gamma_{2k+1} y^{2k+1}. \end{aligned} \quad (10)$$

In [9-12], we have presented a solution of Smale's thirteenth problem [28] proving that the Liénard system (10) with a polynomial of degree $2k + 1$ can have at most k limit cycles and we can conclude now that our results [9-12] agree with the conjecture of [19] on the maximum number of limit cycles for the classical Liénard polynomial system (10). There were some attempts to construct counterexamples to this conjecture, e. g., in [6,7]. But that "counterexamples" were completely wrong.

In [13-16], we have studied the general Liénard polynomial system ($h(x) \equiv 0$)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + a_1 x + \dots + a_{2l} x^{2l}) \\ &\quad + y(\alpha_0 + \alpha_1 x + \dots + \alpha_{2k} x^{2k}). \end{aligned} \quad (11)$$

In [13-15], under some assumptions on the parameters of (11), and in [16], in the general case, we have found the maximum number of limit cycles and their possible distribution for system (11).

We use the obtained results and develop our methods for studying limit cycle bifurcations of polynomial dynamical systems in this paper as well. In Section II, applying canonical systems with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the problem on the maximum number and distribution of limit cycles in an Euler-Lagrange-Liénard type mechanical system. This is related to the solution of Hilbert's sixteenth problem on the maximum number and distribution of limit cycles in planar polynomial dynamical systems [8].

2. Limit Cycle Bifurcations

By means of our bifurcational geometric approach [8-16], we will consider now the Euler-Lagrange-Liénard polynomial system (7). The study of singular points of system (7) will use two index theorems by H. Poincaré; see [2]. The definition of the Poincaré index is the following [2].

Definition 1: Let S be a simple closed curve in the phase plane not passing through a singular point of the system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (12)$$

where $P(x, y)$ and $Q(x, y)$ are continuous functions (for example, polynomials), and M be some point on S . If the point M goes around the curve S in the positive direction (counterclockwise) one time, then the vector coinciding with

the direction of a tangent to the trajectory passing through the point M is rotated through the angle $2\pi j$ ($j = 0, \pm 1, \pm 2, \dots$). The integer j is called the *Poincaré index* of the closed curve S relative to the vector field of system (12) and has the expression

$$j = \frac{1}{2\pi} \oint_S \frac{P dQ - Q dP}{P^2 + Q^2}. \quad (13)$$

According to this definition, the index of a node or a focus, or a center is equal to $+1$ and the index of a saddle is -1 . The following Poincaré index theorems are valid [2].

Theorem 1 (First Poincaré Index Theorem): The indices of singular points in the plane and at infinity sum to $+1$.

Theorem 2 (Second Poincaré Index Theorem): If all singular points are simple, then along an isocline without multiple points lying in a Poincaré hemisphere which is obtained by a stereographic projection of the phase plane, the singular points are distributed so that a saddle is followed by a node or a focus, or a center and vice versa. If two points are separated by the equator of the Poincaré sphere, then a saddle will be followed by a saddle again and a node or a focus, or a center will be followed by a node or a focus, or a center.

Consider system (7) supposing that $a_1^2 + \dots + a_{2l}^2 \neq 0$. Its finite singularities are determined by the algebraic system

$$x(1 + a_1 x + \dots + a_{2l} x^{2l}) = 0, \quad y = 0. \quad (14)$$

This system always has an anti-saddle at the origin and, in general, can have at most $2l + 1$ finite singularities which lie on the x -axis and are distributed so that a saddle (or saddlenode) is followed by a node or a focus, or a center and vice versa [2]. For studying the infinite singularities, the methods applied in [2] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [8] can be used; see [9-16].

Following [8], we will study limit cycle bifurcations of (7) by means of canonical systems containing field rotation parameters of (7) [2,8].

Theorem 3: The Euler-Lagrange-Liénard polynomial system (7) with limit cycles can be reduced to one of the canonical forms:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + a_1 x + \dots + a_{2l} x^{2l}) \\ &\quad + y(\alpha_0 - \beta_1 - \dots - \beta_{2k-1} + \beta_1 x \\ &\quad + \alpha_2 x^2 + \dots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}) \\ &\quad + y^2(c_0 + c_1 x + \dots + c_{2n} x^{2n}) \end{aligned} \quad (15)$$

or

$$\begin{aligned} \dot{x} &= y \equiv P(x, y), \\ \dot{y} &= x(x-1)(1 + b_1 x + \dots + b_{2l-1} x^{2l-1}) \\ &\quad + y(\alpha_0 - \beta_1 - \dots - \beta_{2k-1} + \beta_1 x \\ &\quad + \alpha_2 x^2 + \dots + \beta_{2k-1} x^{2k-1} + \alpha_{2k} x^{2k}) \\ &\quad + y^2(c_0 + c_1 x + \dots + c_{2n} x^{2n}) \equiv Q(x, y), \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(-\beta_{2k-3}-\beta_{2k-1}+\beta_{2k-3}x^{2k-3}+\beta_{2k-1}x^{2k-1}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (20)$$

Then the vector field of (20) is rotated in the opposite directions in each of the half-planes $x < 1$ and $x > 1$. Under decreasing β_{2k-3} , when $\beta_{2k-3} = -\beta_{2k-1}$, the focus at the origin becomes nonrough (weak), changes the character of its stability and generates a stable limit cycle. All of the other foci in the half-plane $x > 1$ will also generate unstable limit cycles for some values of β_{2k-3} after changing the character of their stability. Under further decreasing β_{2k-3} , all of the limit cycles will expand disappearing on separatrix cycles of (20) [2,8].

Denote the limit cycle surrounding the origin by Γ_0 , the domain outside the cycle by D_{01} , the domain inside the cycle by D_{02} and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding this singular point. It is clear that, under decreasing the parameter β_{2k-3} , a semi-stable limit cycle cannot appear in the domain D_{02} , since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [9-16].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_{01} . Suppose it appears in this domain for some values of the parameters $\beta_{2k-1}^* > 0$ and $\beta_{2k-3}^* < 0$. Return to system (17) and change the inputting order for the semi-rotation parameters. Input first the parameter $\beta_{2k-3} < 0$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(-\beta_{2k-3}+\beta_{2k-3}x^{2k-3}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (21)$$

Fix it under $\beta_{2k-3} = \beta_{2k-3}^*$. The vector field of (21) is rotated counterclockwise and the origin turns into a rough unstable focus. Inputting the parameter $\beta_{2k-1} > 0$ into (21), we get again system (20)-(21), the vector field of which is rotated clockwise. Under this rotation, a stable limit cycle Γ_0 will appear from a separatrix cycle for some value of $\beta_{2k-1} > 0$. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing β_{2k-1} to the value β_{2k-1}^* . It follows that there are no values of $\beta_{2k-3}^* < 0$ and $\beta_{2k-1}^* > 0$ for which a semi-stable limit cycle could appear in the domain D_{01} .

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (20) for any values of the parameters β_{2k-3} and β_{2k-1} of different signs. Obviously, if these parameters have the same sign, system (20) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than l limit cycles surrounding the other singularities (foci or nodes) of (20) one by one.

It is clear that inputting the other semi-rotation parameters $\beta_{2k-5}, \dots, \beta_1$ into system (20) will not give us more limit cycles, since all of these parameters are rough with respect to the origin and the other anti-saddles lying in the half-plane $x > 1$. Therefore, the maximum number of limit cycles for the system"

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(-\beta_1-\dots-\beta_{2k-3}-\beta_{2k-1}+\beta_1x+\dots \\ &+ \beta_{2k-3}x^{2k-3}+\beta_{2k-1}x^{2k-1}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}) \end{aligned} \quad (22)$$

is equal to $l+1$ and they surround the anti-saddles (foci or nodes) of (22) one by one.

Suppose that $\beta_1+\dots+\beta_{2k-3}+\beta_{2k-1} > 0$ and input the last rough parameter $\alpha_0 > 0$ into system (22):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\alpha_0-\beta_1-\dots-\beta_{2k-1}+\beta_1x+\dots+\beta_{2k-1}x^{2k-1}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (23)$$

This parameter rotating the vector field of (23) counterclockwise in the whole phase plane also will not give us more limit cycles, but under increasing α_0 , when $\alpha_0 = \beta_1+\dots+\beta_{2k-1}$, we can make the focus at the origin nonrough (weak), after the disappearance of the limit cycle Γ_0 in it. Fix this value of the parameter α_0 ($\alpha_0 = \alpha_0^*$):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x+\dots+\beta_{2k-1}x^{2k-1}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (24)$$

Let us input now successively the other field rotation parameters $\alpha_2, \dots, \alpha_{2k}$ into system (24) beginning again with the parameters at the highest degrees of x and alternating with their signs; see [9-16]. So, begin with the parameter α_{2k} and let $\alpha_{2k} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x+\dots+\beta_{2k-1}x^{2k-1}+\alpha_{2k}x^{2k}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (25)$$

In this case, the vector field of (25) is rotated clockwise in the whole phase plane and the focus at the origin changes the character of its stability generating again a stable limit cycle. The limit cycles surrounding the other singularities of (25) can also still exist. Denote the limit cycle surrounding the origin by Γ_1 , the domain outside the cycle by D_1 and the domain inside the cycle by D_2 . The uniqueness of a limit cycle surrounding the origin (and limit cycles surrounding the other singularities) for system (25) can be proved by contradiction like we have done above for (20); see [9-16].

Let system (25) have the unique limit cycle Γ_1 surrounding the origin and l limit cycles surrounding the other antisaddles of (25). Fix the parameter $\alpha_{2k} < 0$ and input the parameter $\alpha_{2k-2} > 0$ into (25):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x+\dots+\beta_{2k-1}x^{2k-1}+\alpha_{2k-2}x^{2k-2} \\ &+ \alpha_{2k}x^{2k}) + y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (26)$$

Then the vector field of (26) is rotated in the opposite direction (counterclockwise) and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of x changes) generating the second (unstable) limit cycle Γ_2 . The limit cycles surrounding the other singularities of (26) can only disappear in the corresponding foci (because of their roughness) under increasing the parameter α_{2k-2} . Under further increasing α_{2k-2} , the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12} , which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12} ? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter [9-16].

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i.e., that for some values of these parameters, $\alpha_{2k} < 0$ and $\alpha_{2k-2} > 0$, such a semi-stable cycle exists. Return to system (24) again and input first the parameter $\alpha_{2k-2} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x+\dots+\beta_{2k-1}x^{2k-1}+\alpha_{2k-2}x^{2k-2}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (27)$$

This parameter rotates the vector field of (27) counter clock wise preserving the origin as a nonrough stable focus.

Fix this parameter under $\alpha_{2k-2} = \alpha_{2k-2}^*$ and input the parameter $\alpha_{2k} < 0$ into (27) getting again system (26). Since, by our assumption, this system has two limit cycles surrounding the origin for $\alpha_{2k} > \alpha_{2k}^*$, there exists some value of the parameter, α_{2k}^{12} ($\alpha_{2k}^{12} < \alpha_{2k}^* < 0$), for which a semistable limit cycle, Γ_{12} , appears in system (26) and then splits into a stable cycle Γ_1 and an unstable cycle Γ_2 under further decreasing α_{2k} . The formed domain D_2 bounded by the limit cycles Γ_1 , Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing α_{2k} . The distance between the spirals of the domain D_2 will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k} < \alpha_{2k}^{12}$ [9-16].

Thus, there are no such values of the parameters, $\alpha_{2k}^* < 0$ and $\alpha_{2k-2}^* > 0$, for which system (26) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters α_{2k} and α_{2k-2} for which system (26) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (26). Therefore, $l+2$ is the maximum number of limit cycles in system (26).

Suppose that system (26) has two limit cycles, Γ_1 and Γ_2 , surrounding the origin and l limit cycles surrounding the other antisaddles of (26) (this is always possible if $-\alpha_{2k} \gg \alpha_{2k-2} > 0$). Fix the parameters α_{2k}^* , α_{2k-2} and consider a more general system inputting the third parameter, $\alpha_{2k-4} < 0$, into (26):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x+\dots+\beta_{2k-1}x^{2k-1} \\ &+ \alpha_{2k-4}x^{2k-4}+\alpha_{2k-2}x^{2k-2}+\alpha_{2k}x^{2k}) \\ &+ y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (28)$$

For decreasing α_{2k-4} , the vector field of (28) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, Γ_3 . With further decreasing α_{2k-4} , Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23} , which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand disappearing on a separatrix cycle of (28).

Let system (28) have three limit cycles surrounding the origin: Γ_1 , Γ_2 , Γ_3 . Could an additional semi-stable limit cycle appear with decreasing α_{2k-4} after splitting of which system (28) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains after decreasing α_{2k-4} . Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3 . As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha_{2k}^* < 0$, $\alpha_{2k-2}^* > 0$ and $\alpha_{2k-4}^* < 0$ such a semi-stable cycle exists. Return to system (24) again inputting first the parameters $\alpha_{2k-2} > 0$ and $\alpha_{2k-4} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1+b_1x+\dots+b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x+\dots+\beta_{2k-1}x^{2k-1}+\alpha_{2k-4}x^{2k-4} \\ &+ \alpha_{2k}x^{2k}) + y^2(c_0+c_1x+\dots+c_{2n}x^{2n}). \end{aligned} \quad (29)$$

Fix the parameter α_{2k-2} under the value α_{2k-2}^* . With decreasing α_{2k-4} , a separatrix cycle formed around the origin will generate a stable limit cycle Γ_1 . Fix α_{2k-4} under the value α_{2k-4}^* and input the parameter $\alpha_{2k} > 0$ into (29) getting system (28).

Since, by our assumption, (28) has three limit cycles for $\alpha_{2k} > \alpha_{2k}^*$, there exists some value of the parameter α_{2k}^{23} ($\alpha_{2k}^{23} < \alpha_{2k}^* < 0$) for which a semi-stable limit cycle, Γ_{23} , appears in this system and then splits into an unstable cycle Γ_2 and a stable cycle Γ_3 with further decreasing α_{2k} . The formed domain D_3 bounded by the limit cycles Γ_2 , Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there [9-16].

All other combinations of the parameters α_{2k} , α_{2k-2} , and α_{2k-4} are considered in a similar way. It follows that system (28) can have at most $l + 3$ limit cycles.

If we continue the procedure of successive inputting the field rotation parameters, $\alpha_{2k}, \dots, \alpha_2$, into system (24),

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x(x-1)(1 + b_1x + \dots + b_{2l-1}x^{2l-1}) \\ &+ y(\beta_1x + \dots + \beta_{2k-1}x^{2k-1} + \alpha_2x^2 + \dots \\ &+ \alpha_{2k}x^{2k}) + y^2(c_0 + c_1x + \dots + c_{2n}x^{2n}), \end{aligned} \quad (30)$$

it is possible to obtain k limit cycles surrounding the origin and l surrounding one by one the other singularities (foci or nodes) ($-\alpha_{2k} \gg \alpha_{2k-2} \gg -\alpha_{2k-4} \gg \alpha_{2k-6} \gg \dots$).

Then, by means of the parameter $\alpha_0 \neq \beta_1 + \dots + \beta_{2k-1}$ ($\alpha_0 > \alpha_0^*$, if $\alpha_2 < 0$, and $\alpha_0 < \alpha_0^*$, if $\alpha_2 > 0$), we will have the canonical system (16) with an additional limit cycle surrounding the origin and can conclude that this system (i. e., the Euler-Lagrange-Liénard polynomial system (7) as well) has at most $k+l+1$ limit cycles, $k+1$ surrounding the origin and l surrounding one by one the antisaddles (foci or nodes) of (16) (and (7) as well). The theorem is proved.

References

- Agarwal, A., & Ananthkrishnan, N. (2000). Bifurcation analysis for onset and cessation of surge in axial flow compressors. *International Journal of Turbo and Jet Engines*, 17(3), 207-218.
- Bautin, N.N., Leontovich, E.A. (1990). *Methods and Examples of the Qualitative Analysis of Dynamical Systems in a Plane*, Nauka, Moscow.
- Bessa, W. M., Dutra, M. S., & Kreuzer, E. (2008). Depth control of remotely operated underwater vehicles using an adaptive fuzzy sliding mode controller. *Robotics and Autonomous Systems*, 56(8), 670-677.
- Catalano, M. G., Grioli, G., Garabini, M., Belo, F. A. W., Di Basco, A., Tsagarakis, N. G., & Bicchi, A. (2012, May). A Variable Damping module for Variable Impedance Actuation. In ICRA (pp. 2666-2672).
- Chang-Jian, C. W., & Chen, C. O. K. (2009). Nonlinear analysis of a rub-impact rotor supported by turbulent couple stress fluid film journal bearings under quadratic damping. *Nonlinear Dynamics*, 56(3), 297-314.
- P. De Maesschalck, F. Dumortier, Classical Liénard equations of degree $n \geq 6$ can have $[(n-1)/2] + 2$ limit cycles, *J. Differential Equations* 250 (2011), 2162-2176.
- Dumortier, F., Panazzolo, D., & Roussarie, R. (2007). More limit cycles than expected in Liénard equations. *Proceedings of the American Mathematical Society*, 135(6), 1895-1904.
- Gaiko, V. (2013). Global bifurcation theory and Hilbert's sixteenth problem (Vol. 562). *Springer Science & Business Media*.
- V. A. Gaiko, Limit cycles of Liénard-type dynamical systems, *Cubo 10* (2008), 115-132.
- Gaiko, V. A. (2009). On the geometry of polynomial dynamical systems. *Journal of Mathematical Sciences*, 157(3), 400-412.
- Gaiko, V. A. (2011). The geometry of limit cycle bifurcations in polynomial dynamical systems. *Discrete Contin. Dyn. Syst. Suppl*, 5, 447-456.
- V. A. Gaiko, On limit cycles surrounding a singular point, *Differ. Equ. Dyn. Syst.* 20 (2012), 329-337.
- Gaiko, V. A. (2012). The applied geometry of a general Liénard polynomial system. *Applied Mathematics Letters*, 25(12), 2327-2331.
- Gaiko, V. A. (2012). Limit cycle bifurcations of a general Liénard system with polynomial restoring and damping functions. *International Journal of Dynamical Systems and Differential Equations*, 4(3), 242-254.
- Gaiko, V. A. (2014). Limit cycle bifurcations of a special Liénard polynomial system. *Adv. Dyn. Syst. Appl*, 9(1), 109-123.
- Gaiko, V. A. (2015). Maximum number and distribution of limit cycles in the general Liénard polynomial system. *Adv. Dyn. Syst. Appl*, 10(2), 177-188.
- Han, M., Tian, Y., & Yu, P. (2013). Small-amplitude limit cycles of polynomial Liénard systems. *Science China Mathematics*, 56(8), 1543-1556.
- Laalej, H., Lang, Z. Q., Daley, S., Zazas, I., Billings, S. A., & Tomlinson, G. R. (2012). Application of non-linear damping to vibration isolation: an experimental study. *Nonlinear Dynamics*, 69(1), 409-421.
- A. Lins, W. de Melo, C. C. Pugh, On Liénard's equation, *Lecture Notes in Mathematics* 597 (1977), Springer, Berlin, 335-357.
- Lloyd, N. G. (1987, November). Liénard systems with several limit cycles. In *Mathematical Proceedings of the Cambridge Philosophical Society* (Vol. 102, No. 3, pp. 565-572). Cambridge University Press.
- Owens, D. B., Capone, F. J., Hall, R. M., Brandon, J. M., & Chambers, J. R. (2004). Transonic free-to-roll analysis of abrupt wing stall on military aircraft. *Journal of aircraft*, 41(3), 474-484.
- G. S. Rychkov, The maximal number of limit cycles of the system $\dot{y} = -x$, $\dot{x} = y - \sum_{i=0}^2 a_i x^{2i+1}$ is equal to two, *Differ. Equ.* 11 (1975), 301-302.
- Savin, S., Golousov, S., Khusainov, R., Balakhnov, O., & Klimchik, A. (2019, July). Control system design for two link robot arm with macepa 2.0 variable stiffness actuators. In *2019 IEEE/ASME International Conference on Advanced Intelligent Mechatronics (AIM)* (pp. 624-628). IEEE.
- Savin, S., Khusainov, R., & Klimchik, A. (2019). Control of actuators with linearized variable stiffness. *IFAC-PapersOnLine*, 52(13), 713-718.

25. Slight, T. J., Romeira, B., Wang, L., Figueiredo, J. M., Wasige, E., & Ironside, C. N. (2008). A Liénard oscillator resonant tunnelling diode-laser diode hybrid integrated circuit: model and experiment. *IEEE journal of quantum electronics*, 44(12), 1158-1163.
26. Shiriaev, A., Perram, J. W., & Canudas-de-Wit, C. (2005). Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on Automatic Control*, 50(8), 1164-1176.
27. Shiriaev, A., Robertsson, A., Perram, J., & Sandberg, A. (2006). Periodic motion planning for virtually constrained Euler–Lagrange systems. *Systems & control letters*, 55(11), 900-907.
28. Smale, S. (1998). Mathematical problems for the next century. *The mathematical intelligencer*, 20(2), 7-15.