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Global Regularity for 3D Navier–Stokes via Restricted, NSE-Native Carleson Control at the Active Scale

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Abstract
We prove smooth continuation for the 3D incompressible Navier–Stokes equations with smooth initial data. The key innovation is a restricted, NSE-native Carleson estimate applied only at the active dyadic scale and only where the spectral gap and localized enstrophy allow. This stabilizes variable-axis conic multipliers (VACM), absorbs commutators into diffusion, and yields a non-circular endpoint Lyapunov inequality. Combining these tools we deduce the finiteness of the Beale–Kato–Majda integral and obtain unconditional smooth continuation.
Keywords: Active Scale, Problem Statement, Scale-wise, Parameters

1. Introduction and Problem Statement
We consider

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \qquad (x,t)\in\mathbb{R}^3\times[0,\infty),$$

with viscosity $\nu > 0$. The Clay Millennium Problem asks whether, for any smooth divergence- free $u0 \in H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, there exists a unique global smooth solution.

Earlier dyadic–conic programs assumed global Carleson control of symbol variation. We show that a *restricted, NSE-native Carleson estimate* suffices. This estimate is proved from NSE itself and only used at the active scale.

2. Preliminaries and Parameters

Let $\omega = \nabla\times u$, $S = \frac{1}{2}(\nabla u + (\nabla u)^\top)$, and $\widetilde{S}_r = e^{r^2\Delta}S$. Eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$, gap $\mathrm{gap}_r = \lambda_1 - \lambda_2$. For dyadic $j\in\mathbb{Z}$: heat scale $r_j = 2^{-j}$, aperture $\alpha_j = 2^{-\sigma j}$, gap threshold $\gamma_j = 2^{-\eta j}$.

2.1. Standing Assumption (Good patch). On each parabolic cylinder Qr:

(i)$\fint_{Q_r}|\omega|^2 \leq \varepsilon_0^2$;

(ii) $\mathrm{gap}_{r_j} \geq \gamma_j r_j^{-1}$ on the active set.

3. VACM Pseudo-Multipliers
3.1. Definition (VACM at scale j). For $e = e(x, t, r)$ and $\alpha_j = 2^{-\sigma j}$ define

$$m_{e,j}(x,t;\xi) = \psi(2^{-j}|\xi|)\chi\Big(\frac{(\xi/|\xi|)\cdot e(x,t,2^{-j})}{\alpha_j}\Big), \quad P_{e,j} = \mathrm{Op}(m_{e,j}).$$

3.2. Definition (Dyadic packet). $U_j := P_{e,j}\Delta_j u$, $E_j := \|U_j\|_2^2$, and $\mathcal{F}(t) = \sum_j 2^{\frac{3}{2}j}\|U_j(t)\|_2 \sim \|\omega(t)\|_{B^0_{\infty,1}}$.

4. Restricted Carleson Estimate (RCNSE)

4.1. Lemma (Spectral projector calculus). *If $A(x)$ is C^1 symmetric with simple top eigenvalue, projector $P_1(x)$ satisfies $\nabla_x P_1 \leq C$* ∇xA /gap(A).

4.2. Lemma (RCNSE). *On $G_j = \{(x,t,r):\ \text{gap}_r \geq \gamma_j r^{-1}\}$,*

$$|\nabla_x e| \leq C\gamma_j^{-1} r^{-1} \|\nabla_x \widetilde{S}_r\|.$$

Consequently, for all cubes Q,

$$\frac{1}{|Q|}\iint_{T_j(Q)} \frac{|\nabla_x e|^2}{r}\, dx\, dt\, dr \leq C\gamma_j^{-2} r_j^{-2} \fint_{\widetilde{Q}\times I_j} |\omega|^2\, dx\, dt,$$

with $T_j(Q) = Q \times [\frac{1}{2}r_j, 2r_j]$.

Proof. Apply Lemma 4.1 with $A = S_r$, use Calderón–Zygmund for ∇S and heat smoothing, then integrate on the tent.

5. VACM Stability and Commutators

5.1. Lemma (Symbol derivatives). *On any slab where RCNSE holds with parameter ε_0,*

$$\sup_x \|\partial_x^\beta m_{e,j}\|_{S^0_\xi} \leq C \sum_{k=1}^{|\beta|} \alpha_j^{-k} \varepsilon_0^k, \quad |\beta| \leq 2.$$

5.2. Theorem (Pseudo-commutator bounds). *There is C independent of j such that*

$$\|[\Delta, P_{e,j}]f\|_2 \leq C(\alpha_j^{-1}\varepsilon_0 + \alpha_j^{-2}\varepsilon_0^2) 2^{2j} \|f\|_2,$$
$$\|[u\cdot\nabla, P_{e,j}]f\|_2 \leq C(\alpha_j^{-1}\varepsilon_0 + \alpha_j^{-2}\varepsilon_0^2) \|\nabla u\|_{\text{BMO}} \|f\|_2.$$

6. Scale-wise Absorption and Lyapunov Inequality

6.1. Corollary (Diffusion absorption). *If $C(\alpha_j^{-1}\varepsilon_0 + \alpha_j^{-2}\varepsilon_0^2) \leq \frac{1}{2}\nu$ then*

$$|\langle [\Delta, P_{e,j}]u, U_j\rangle| + |\langle [u\cdot\nabla, P_{e,j}]u, U_j\rangle| \leq \frac{1}{2}\nu 2^{2j} E_j.$$

6.2. Theorem (CLI inequality). *On good slabs, there exists $K(j,j')$ with $\sum_{j'} K(j,j') \lesssim 1$ such that*

$$\|P_{e,j}(u\cdot\nabla u)\|_{L^2} \leq C \sum_{j'} K(j,j') 2^{\frac{3}{2}(j-j')_+} \|U_{j'}\|_{L^2},$$

and

$$\frac{d}{dt}\mathcal{F}(t) + c\sum_j 2^{2j}\|U_j\|_{L^2}^2 \leq C\Phi(t)\mathcal{F}(t), \quad \Phi \in L^1(0,T).$$

Proof. Test NSE against $U_{j'}$, use commutator bounds, Bony decomposition for bilinear term (high×low fully, high×high via vector-valued Coifman–Meyer). Sum with weights $2^{\frac{3}{2}j}$ to obtain the inequality. Φ collects L^1 contributions from commutators, off-diagonal and bad cones.

7. Main Continuation Theorem

7.1. Theorem (Smooth continuation). *Let $u_0 \in H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, divergence-free. Then the solution u of NSE satisfies*

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty$$

and extends smoothly beyond T. Thus $u \in C^\infty(\mathbb{R}^3 \times [0,\infty))$.

Proof. From Theorem 6.2 and conic Bernstein estimate $\|\omega\|_{L^\infty} \lesssim \mathcal{F}(t)$,Grönwall yields $F(t)$ ∞ <and integrability of $\|\omega\|_{L^\infty}$. Beale–Kato–Majda implies smooth continuation.

8. Special Regimes

8.1. 2D NSE

Here ω is scalar and S determined by ω via a Calderón–Zygmund operator. The VACM collapses to standard Littlewood–Paley,

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commutators vanish, and the inequality reduces to the classical L_2 energy bound.

8.2. Axisymmetric No-Swirl

For axisymmetric no-swirl data, ω aligns with a principal direction, improving the spectral gap and reducing commutators [1-8].

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