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Multiscale Stochastic Volatility Problem with Wavelet Risk Premium

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Abstract

In this paper, the mean variance portfolio with multiscale stochastic volatility using the wavelet risk premium was considered. A seemingly wavelet function which was used to investigate the multiscale stochastic volatility (MSSV) used in stock price model was derived. Two types of volatility, namely; a fast -moving one and a slowly-moving one were considered using the stochastic dynamic programming principle and Hamilton-Jacobi-Bellman equation approach. The optimal investment strategy, the value function was obtained.

Keywords: Mean-variance Portfolio Selection, Multiscale Stochastic Volatility, Stochastic Control, Probability, Wavelets

1. Introduction

Wavelet method, as a time-frequency analysis method has been applied in various fields to analyze a wide range of signals covering all aspect of life. It had capacity to provide both time and frequency domains information, wavelet analysis is mainly for time-frequency analysis of signals, signal compression, signal denoising, singularity analysis and features extraction. Wavelet analysis is very useful for analyzing physiological systems because, as opposed to most classical signal analysis approaches, it provides the means to detect and analyze non stationarity in signals. Wavelet methods have been successfully used for solving option pricing problems, see e.g. [4, 5, 6, 12]. Also the mean variance portfolio selection problem with multiscale stochastic volatility was also proposed [2]. The presence of volatility factors is well documented in the literature using underlying returns data (see [1,3,7, 8,9,11,13,14,]. Mean-variance theory is an important model of investments based on decision theory. It is the simplest model of investments that is sufficiently rich to be directly useful in applied problems. In this paper we focus on analyzing the mean variance portfolio with multiscale stochastic volatility using the wavelet risk premium.

1.1. Model Formulation

The classical Black Scholes model based on Ito's process for derivative or (spot asset) prices S follows a geometric Brownian Motion, and is given by the Stochastic differential equation (SDE):

$$dS = \mu S dt + \sigma S dW \qquad (2.1.1)$$

Where S = security price μ = constant drift
σ = constant volatility
W = standard wiener process

In finance, the Black Scholes Partial differential equation (BSPDE) model can be written in the form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \qquad (2.1.2)$$

using the boundary conditions for

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$$W(x\ ,\ \tau):$$

$$W_0\left(x_\tau\right) = W\left(x_\tau,\ 0\right) = e^{\alpha x_\tau} V(x_\tau,\ 0)$$
$$\left(e^{(\alpha+1)x_\tau}\ -\ e^{\alpha x_\tau}\right)^+ \ = \left(e^{\beta x_\tau} -\ e^{\alpha x_\tau}\right)^+ \qquad (2.1.3)$$

In [18] approach we eliminate S and S^2 terms in equation (2.1.2), and have the following equation after transformations.

$$-\frac{\partial U}{\partial \tau} + (K-1)\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - KU = 0, \qquad (2.1.4)$$

where

$$K = \frac{2\tau}{\sigma^2}, \qquad (2.1.5)$$

given the boundary condition: $V(S_T, T) = (S_T - K)^+$.

Let

$$U(x,\ \tau) = \frac{1}{K}V\ \ (S,\ t) = \frac{1}{K}V\left(Ke^x,\ T - \frac{2\tau}{\sigma^2}\right) \qquad (2.1.6)$$

be given. When $t = T$, $S_t = S_T$,

$$\begin{aligned} x &= ln\frac{S_T}{K} \\ i.e\quad x_T &= ln\frac{S_T}{K}, \end{aligned}$$

and $\tau = 0$ $\left(\tau = \frac{\sigma^2(T-t)}{2}\right)$. Hence, the boundary condition for V:

$$\begin{aligned} U_0(x_T) = U_0\left(x_{T,}\ 0\right) &= \frac{1}{K}\ V(S_T - K)^+ \\ &= \frac{1}{K}(Ke^x - K)^+ \end{aligned}$$
$$U_0 x_T, 0) = (e^{x_T} - 1)^+. \qquad (2.1.7)$$

We transform again:

$$W\left(x\ ,\tau\right) = \ e^{\alpha x + \beta^2 \tau}\ U(x,\tau), \qquad (2.1.8)$$

where

$$\alpha = \frac{1}{2}\left(K-1\right), \qquad (2.1.9)$$
$$\beta = \frac{1}{2}(K+1), \qquad (2.1.10)$$

and

$$U\left(x,\tau\right) = W(x,\tau)e^{-\alpha x - \beta^2 \tau} \qquad (2.1.11)$$

which converts (2.1.3) into the heat equation as follows:

$$\begin{aligned} U\left(x,\tau\right) &= W\left(x,\tau\right)e^{-\alpha x - \beta^2\tau} \\ \frac{\partial U}{\partial \tau} &= e^{-\alpha x - \beta^2\tau}\left[\frac{\partial W}{\partial \tau} - W(x,\tau)\beta^2\right] \\ \frac{\partial U}{\partial x} &= e^{-\alpha x - \beta^2\tau}\left[\frac{\partial W}{\partial x} - \alpha W(x,\tau)\right] \end{aligned}$$

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$$\frac{\partial^2 U}{\partial x^2} = e^{-\alpha x-\beta^2\tau}\left[\alpha^2 W\left(x,\tau\right) - 2\alpha\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2}\right]. \qquad (2.1.12)$$

Substituting (2.1.3) into (2.1.12) yields,

$$\begin{gathered}\beta^2 W\left(x,t\right) - \tfrac{\partial W}{\partial \tau} + \left(K-1\right)\left[-\alpha W\left(x,\tau\right) + \tfrac{\partial W}{\partial x}\right] + \alpha^2 W\left(x,\tau\right)\\ -2\alpha\tfrac{\partial^2 W}{\partial x^2} + \tfrac{\partial^2 W}{\partial x^2} - KW\left(x,\tau\right) = 0.\end{gathered}$$

This reduces to the heat equation:

$$\frac{\partial W}{\partial \tau} = K\frac{\partial^2 W}{\partial x^2}. \qquad (2.1.13)$$

The solution of equation (2.1.13) is by the method of variation (see [15, 16]).

1.2. Formulation of Wavelet Function as Risk Premium and Investment Strategy

Lemma 2.1. Let x_k be a Gaussian centred process with stationary increment and the the wavelet risk premium., $\pi(t)$ denote the amount invested in the (zero) bond at time t and $W(t)$ be a one dimensional standard Brownian motion and $\overline{W}$ its Fourier transform, σ is the stochastic volatility with $\sigma > 0$ then the Wavelet function and the investment strategy are respectively;

$$U\left(x,t\right) = x_k\overline{W}e^{\frac{-\sigma}{2}^2\pi^2(t)t} \qquad (2.1.14a)$$

and

$$\pi\left(t\right) = \frac{kf}{\pi(v^2 - f^2k^2}e^{-rt}. \qquad (2.1.14b)$$

Proof:

The solution of (2.1.13) by method of variation of parameter given as:

$$\overline{U}(x,t) = \frac{2u_0}{\pi}\sum_{j=1}^{N}\left[\left(\frac{(-1)^{j+1}L+1}{j}\right)e^{-j^2\pi^2kt/l^2}\right]\sin\frac{j\pi k}{L}. \qquad (2.1.15)$$

Using $W(x,\tau) = e^{\alpha x+\beta^2\tau}\,U(x,\tau)$, (2.1.15) becomes

$$U\ \ (x,t) = \frac{2u_0}{\pi}\sum_{j=1}^{N}\left[\left((-1)^{j+1}\frac{L+1}{j}\right)e^{\frac{-\sigma}{2}^2 j^2\pi^2 t/L}\right]\sin\frac{j\pi x}{L}\ e^{-\alpha x-\beta\tau},$$

but $r = \alpha x - \beta\tau$

$$\begin{aligned}U\ \ (x,t) &= \frac{2u_0}{\pi}\sum_{j=1}^{N}\left[\left((-1)^{j+1}\frac{L+1}{j}\right)e^{\frac{-\sigma}{2}^2 j^2\pi^2 t/L}\right]\sin\frac{j\pi x}{L}e^{-rt}\\ &= \sum_{j=1}^{N}x_k\left(j\right)\overline{w}\left(j\right)e^{\frac{-\sigma}{2}^2 j^2\pi^2 t/L} \qquad (2.1.16)\end{aligned}$$

where

$$x_k\left(j\right) = \sin\frac{j\pi x}{L}e^{-rt} \qquad (2.1.17)$$

$x_k\,(j)$ is called the Wavelet risk premium.

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$$\overline{w}_k\left(j\right) = \frac{2u_0}{\pi}\left[\frac{\left(-1\right)^{j+1}L+1}{j}\right]. \tag{2.1.18}$$

Now let L be the Fourier wavelength such that, $L = \frac{2\pi j}{m+1/2}$ and m = *paul order*. Then (2.1.18) gives:

$$\begin{aligned} \overline{W}(j) &= \frac{2u_0}{\pi}\left[\left(\ \left(-1\right)^{j+1}\frac{2\pi j}{m+1/2}+1\right)\right] \\ &= \frac{2u_0}{\pi}\left[\frac{\left(-1\right)^{j+1}\left(2\pi j\right)+\left(m+1/2\right)}{(m+1/2)j}\right] \\ &= \begin{cases} \frac{2u_0}{\pi}\left(\frac{2\pi+m+1/2}{m+1/2}\right) \quad , \quad j \leq 1 \\ -\frac{2u_0}{\pi}\left(\frac{2\pi+m+1/2}{m+1/2}\right) \quad , \quad j > 1. \end{cases} \end{aligned} \tag{2.1.19}$$

Now $u_0 = \frac{\pi}{2L}$, implies $L = \frac{\pi}{2u_0}$, so that (2.1.16) becomes

$$U\left(x,t\right) = \frac{1}{L}\sum_{j=1}^{N} x_k\left(j\right)\overline{W}(j)e^{\frac{-\sigma}{2}^2 j^2\pi^2 t/L}, \tag{2.1.20}$$

which is equivalent to that in [17]. By Parserval's theorem for wavelet analysis,

$$\sigma^2 = \frac{m+1/2}{CN}\sum_{n=0}^{N=1}\sum_{j=0}^{j=}\left|\frac{u_{t(x_j)}}{x_j}\right|^2. \tag{2.1.21}$$

For $j = 1 = L$, (2.1.20) becomes

$$U\left(x,t\right) = x_k\overline{W}e^{\frac{-\sigma}{2}^2\pi^2 t}. \tag{2.1.22}$$

This equation is called the wavelet function. Furthermore for the investment strategy one has (by 2.1.19),

$$\begin{aligned} \pi\left(t\right) &= \int_{-\infty}^{8} e^{-rt}\sin\left(m+1/2\right)e^{2\pi iFx}\ dx\ \ ktt \\ &= e^{-rt}\int_{-\infty}^{8} f(x)e^{iw_1x}dx \ \ , \qquad w_1 = 2\pi F \\ &= \int_{-\infty}^{\infty} e^{-rt}\left[\frac{1}{2\pi}\int_{-\infty}^{8} f(x)e^{iw_1x}dx\right]e^{iwx}dw \\ &= \int_{-\infty}^{\infty}\frac{e^{-rt}}{2i}\left(e^{2\pi(iF-kf)x} - e^{2\pi(kf+iF)x}\right)dx \\ &= \frac{e^{-rt}}{2i}\int\left(e^{-2\pi(kf+F)x} - e^{2\pi(kf+iF)x}\right)dx. \end{aligned}$$

Putting $a = 2\pi\left(kf + iF\right)$, we obtain

$$f\left(x\right) = \frac{e^{-rt}}{2i}\int\left(e^{-ax} - e^{ax}\right)dx.$$

But

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$$\mathrm{e}^{ax} \;=\; 1 + \; iax \; + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \ldots$$

and

$$\mathrm{e}^{-ax} \;=\; 1 - iax + \frac{(ax)^2}{2!} - \frac{(ax)^3}{3!} + \ldots.$$

So,

$$e^{-ax} - e^{ax} = -2iax - \frac{(ax)^3}{3} - \frac{(ax)^5}{5!} + \ldots$$

$$\begin{aligned}
\tfrac{e^{-rt}}{2i}\int_{-\infty}^{8} f(x)\, e^{ivx}\left(e^{2k\pi ifx} - e^{-2k\pi ifx}\right)dx &= \tfrac{e^{-rt}}{2i}\int_{-\infty}^{8}\left(e^{(v+2\pi if)ix} - e^{(v-2k\pi fx)ix}\right)dx \\
&= \tfrac{e^{-rt}}{2i}\left[\tfrac{e^{(v+2\pi f)ix}}{(v+2\pi f)i} - \tfrac{e^{(v-2k\pi f)ix}}{(v-2k\pi f)i}\right]_{-\infty}^{8} \\
&= e^{-rt}\tfrac{(-v+2\pi kf+v+2\pi fk)}{v^2-2vk\pi f+2vk\pi f-4\pi^{2f^2}k^2} \\
&= \tfrac{4\pi kf}{4\pi^2v^2-4\pi^2f^2k^2}e^{-rt} \\
&= \tfrac{4k\pi f}{4\pi^2(v^2-f^2k^2}e^{-rt} \\
&= \tfrac{kf}{\pi(v^2-f^2k^2}e^{-rt}.
\end{aligned}$$

2. Model

The financial market consists of one risk-free asset and one risky asset. The price of the risk-free asset is governed by the following differential equation

$$dB(t) = B(t)\overline{r}(t)dt, \qquad B(0) = b_0 > 0$$

b_0 is the initial price of the risk-free assets , d represents the differential operators and short rate $\overline{r}(t)$ satisfies the differential equation

$$d\overline{r}\,(t) = \left(\overline{a}\,(t) + b\psi(t)\right)dt + bdw\,(t)\,, \qquad \overline{r}\,(0) = \overline{r_0}.$$

The risk premium $\psi(t)$ is a wavelet and is a continuous function.

The risk Premium is given as

$$let\ x_t(j) = \psi\ \ (j) = \ \sin\frac{j\pi x}{L}e^{-rt}$$

$\pi(t)$ is the wealth at t. $w(t)$ is the standard Brownian motion and $\overline{a}\ (t) = \theta\ (t)\ - ar(t)$ is a stochastic process related to $r\ (t)$ were $a > 0$, and θ is a deterministic and continuously differentiable function [5]. The other asset is a (zero) bond whose price process is modeled as

$$\begin{cases}
\quad dS\,(t) = S\,(t)\left[\left(\overline{r}\,(t) + \ sin\frac{j\pi x}{L}e^{-rt}\mu\,(y\,(t)\,, z\,(t))\right)dt + \beta\,(y\,(t)\,, z\,(t))\,dW^0\,(t)\right], \\
\quad dy\,(t) = \frac{1}{\xi}b(y\,(t)\,dt + \frac{1}{\sqrt{\xi}}a\,(y\,(t))\,dW^1\,(t)\,, \\
\quad dz\,(t) = \delta c\,(z\,(t))\,dt + \sqrt{\delta}\,g\,(z\,(t))\,dW^2\,(t)\,, \\
S\,(0) = s_0 = 0,\ \ y\,(0) = y_0\ ,\ \ z\,(0) = z_0,
\end{cases}$$

were two timescale factors is applied in our stochastic volatility, which means we use $\sigma\ (y_t z_t)$ instead of $\sigma\ (t)$. s_0 , y_0 and z_0 are respectively the initial price and initial volatilities of the risky asset. $\mu\ (y\ (t)\ , z\ (t))$ and $\beta\ (y\ (t)\ , z\ (t))$ are respectively the appreciation rate and volatility rate of the risky assets price. As described in the research work by [6], the dynamics of $y(t)$

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and $z(t)$ respectively shows the fast and slow variation of volatility with very small values of parameters ξ, and δ. Besides, we assume that the process $y(t) = y'(1/\xi)$ in distribution where $y'(t)$ is an ergodic diffusion process with unique invariant distribution ϕ, independent of ξ. As described in [12], we denote $\langle . \rangle$ as the invariant expectation with respect to ϕ.

$$\langle g \rangle = \int g(y)\,\phi\ (dy).$$

The standard Brownian motion, $w^0(t)$, $w^1(t)$ and $w^2(t)$ are correlated with

$$Cov\,(w^0(t),\ w^1(t)) = \rho_1,\quad Cov(w^0,\ w^2(t)) = \rho_2,\ Cov\,(w^1(t),\ w^2(t)) = \rho_3\ \ where$$
$$\text{-}1<\rho_1<1\ ,\ \text{-}1<\rho_2<1\ ,\ \text{-}1<\rho_{12}<1\ \text{ and }\ 1+\ 2\rho_1\rho_2\rho_{12}\text{-}\rho_1^2-\rho_2^2-\rho_{12}^2 >$$

0,to ensure positive definiteness of the covariance matrix of the three Brownian motions Let $\pi(t)$ denote the amount invested in the (zero) bond at time t, ($t \in [0, T]$), and $\pi(t)$ the wealth at time t corresponding to investment strategy π. Then the wealth process satisfies the following stochastic differential equation:

$$\begin{aligned} dX^\pi &= \pi(t)((r(t) + sin\tfrac{j\pi x}{L}e^{-rt}\alpha(y(t), z(t))dt + \beta(y(t)), z(t)dw^0(t)] \\ &+ (X^\pi(t) - \pi(t))r(t)dt \\ &= \left[\pi\,(t)\,sin\tfrac{j\pi x}{L}e^{-rt}\alpha\,(y\,(t)\,,\ z\,(t)) + X^\pi\,(t)\,r\,(t)\right] dt\ + \pi\,(t)\,\beta\,(y\,(t)\,,\ z\,(t))\,dw(t) \\ &\qquad (3.1.1) \end{aligned}$$

with initial condition $X(0) = x_0$. The following Mean-variance portfolio selection problem will be considered in the sequel.

$$P\,(w) = \begin{cases} \min\limits_{\pi(.)\epsilon\Pi(0,x_0)} E[X^\pi\,(T) - w]^2, \\ \\ s.e\ E\,(X^\pi(T)) = w \end{cases}$$

where w is a premium return level and $\pi(0, x_0)$ denotes the set of all admissible controls defined as in [2].

By convex optimization theory, problem $P(w)$ can be solved via the following optimal stochastic control problem with a Lagrange multiplier 2λ.

$$PL1\,(\lambda, w)\ \min_{\pi(.)\epsilon\Pi(0,x_0)} \left\{\mathrm{E}[X^\pi\,(T) - w]^2 - 2\lambda\,[E\,(X^\pi\,(T) - w)]\right\}$$

The relationship between the optimal solutions of these two problems is concluded in the following lemma (see [10]).

Lemma 3.1. Denote by $\Gamma(\lambda)$ and $\widehat{\pi}\,(\lambda, t, x\,(t)\,, y\,(t)\,, z\,(t)\,, t\epsilon\,[0, T])$,respectively, the optimal value and the optimal strategy of problem $PL1(\lambda\,,w)$.

Then the optimal value and the optimal strategy of problem $P(w)$ are

$$\operatorname*{Sup}_{\lambda\epsilon R}\Gamma\,(\lambda)\quad and\quad \{\widehat{\pi}\,(\lambda, t, x\,(t)\,, y\,(t)\,, z\,(t)\,, t\epsilon\,[0, T])\}$$

respectively, where $\lambda^* = arg\mathrm{Sup}_{\lambda\epsilon R}\,\Gamma\,(\lambda)$.

The objective function of problem $PL1(\ \lambda,w)$ can be rewritten as $E[X^\pi\,(T) - (\lambda + w)]^2 - \lambda^2$ the solution of problem $PL1(\ ,w)$ is equivalent to that of the problem

$$PL2\,(\lambda, w)\ \min_{\pi(.)\epsilon\Pi(0,x_0)} \mathrm{E}[X^\pi\,(T) - w]^2$$

For problem $PL2\ (\lambda,w)$ we define the value function,

$$V\,(t,\ x\ ,y\ ,z,\ r) = \min_{\pi(.)\epsilon\Pi(t,x,y,z)} \mathrm{E}\left[(X^\pi\,(T) - (\lambda + w)^2 | X\,(t) = x, z\,(t) = z, y\,(t) = y\right]$$

Using Hamilton-Jacobi-Bellman's Optimality principle

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$$V(t,\ x,y,r) = \min_{\pi(.)\epsilon(t,x,)} E\left[V(t+h), X^{\pi}(t+h), r(t+h)\right] \ \forall h>0$$

Define an operator,

$$\begin{aligned} A^{\pi}V(t,x,y,z,r) = {} & V_t + V_x\left[\pi(t)\, sin\tfrac{j\pi x}{L}e^{-rt}\mu(y,z) + X(t)\,r(t)\right] + \tfrac{b(y)}{\xi}V_y + \delta C(z)V_z \\ + {} & \tfrac{1}{2}V_{xx}\pi^2\beta^2(y,z) + \tfrac{1}{2\xi}V_{yy}\mu^2(y) + \tfrac{1}{2}g^2(z)\,\delta V_{zz} + \tfrac{\pi(t)\beta(y,z)\mu(y)P_1}{\sqrt{\xi}}V_{xy} \\ + {} & \pi(t)\,\beta(y,z)\,g(z)\,P_2\sqrt{\sigma V_Z} + \sqrt{\tfrac{\delta}{\xi}}\mu(y)\,g(z)\,P_{1,2}V_{yz} + \tfrac{b}{\sqrt{\xi}}a(y(t))\,V_{yr} \\ + {} & b\sqrt{\delta}g(z(t))\,V_{zr} + b\pi(t)\,\beta(y,z)\,V_{rx} + \tfrac{1}{2}V_{rr}b^2 + V_r\left(\overline{a}(t) + sin\tfrac{j\pi x}{L}e^{-rt}\right) \end{aligned} \qquad (3.1.2)$$

Using the Ito formular the following equation is obtained.

$$V(t+h, X^{\pi}(t+h), z(t+h), y(t+h), r(t+h))$$

$$= V(t,x,y,z,h) + \int_t^{t+h} A^{\pi(s)}V(s, X(S)\,y(s), z(s)\ \ r(s))\,ds$$

$$+ \int_t^{t+h}(bV_r + V_x\pi(s)\,\beta(y(s), z(s))dw.(s) \qquad (3.1.3)$$

According to [11],if $E_{t,x}\left[\left(\int_t^{t+h}(bV_r + V_x\pi(s)\,\beta(y(s), z(s))\right)^2 ds\right] < +\infty$, then $\left(\int_t^{t+h}(bV_r + V_x\pi(s)\,\beta(y(s), z(s))dw(s)\right)$ is a martigale.therefore when $E_{t,x}\left[\int(bV_r + V_x\pi(s)\,\beta(y(s)\,z(s))\right] < +\infty$ substituting (3.1.3) into (3.1.2), yields the HJB equation of $V(t,x,y,z,r)$as follows.

$$\begin{aligned} & V_t + V_x xr + \tfrac{b(y)}{\xi}V_y + \delta C(z)V_z + \tfrac{1}{2\xi}a^2(y)V_{yy} + \tfrac{1}{2}g^2\delta V_{zz} + \sqrt{\tfrac{\delta}{\xi}}a(y)g(z)p_{12}V_{yz} \\ & + \tfrac{b}{\sqrt{\xi}}a(y(t))V_{yr} + b\sqrt{\delta}g(z)V_{zr} + \tfrac{1}{2}b^2V_{rr} + (\overline{a}(t) + bsin\tfrac{j\pi x}{L}e^{-rt})V_r \\ & + \min\nolimits_{\pi(t)}\left\{\tfrac{1}{2}\pi^2\beta^2(y,z)V_{xx} + \tfrac{\pi\beta(y,z)a(y)p_1}{\sqrt{\xi}}V_{xy}\right. \\ & \left. + \pi(t)\beta(y,z)g(z)p_2\sqrt{\delta}V_{xz} + b\pi(t)\beta(y,z)V_{rx} + \pi(t)sin\tfrac{j\pi x}{L}e^{-rt}\beta(y,z)V_x\right\} \\ & = 0 \end{aligned} \qquad (3.1.4)$$

With the boundary condition$(V, x, y, z, r) = (xyz - (\lambda + w))^2$. We derive the following theorem in order to know the contribution of the HJB equation (3.1.4) to derive the optimal strategy and the value function.

Theorem 3.2. suppose that $v(x, y, z, r) \in C^{1,2,3}([0, T]\times 0)$ where $0 \sqsubseteq R^2$ satisfies
(i) $v(t, x, y, z\ r)$ solves (3.1.4) with boundary condition;
(ii) for any admissible control π (.)and its corresponding wealth process,

$$E_{t,x}\left[\left(\int_t^{t+h}(bV_r + V_x\pi(s)\,\beta(y(s), z(s))\right)^2 ds\right] < +\infty, \forall t \in [0, .],\ h>0$$

(iii) For all sequences of stopping times $(\tau_n : 0 \le \tau \le T)_{n\in N}$ and any admissible strategy $\pi(.) \epsilon \Pi(0, x_0)$, the sequence $\{v(\tau_n, X^{\pi}(\tau_n))\}_{n\epsilon N}$ is uniformly integrable.

Then we have
a) $v(t, x, y, z.r) \le V(t, x, y, z, r)$;
b) if there exists an admissible strategy $\widehat{\pi}(t) \in argmin\, A^{\pi}v(t, X_{\pi}(t), r)$, then

$$v(t, x, y, z.r) = V(t, x, y, z, r)\,.$$

We are going to work to deduce the optimal strategy and the value function of problem $PL2(\lambda,w)$.

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2.1. Optimal Solution

Assume that v is a solution of HJB (3.1.4) when $V_{xx}>0, V_{zz}, V_{yy}>0$ the optimal strategy of problem $PL2(\lambda,w)$ is

$$\widehat{\pi}=-\frac{\left(a\left(y\right)P_1V_{xy}+g\left(z\right)p_2\sqrt{\delta}V_{xz}+bV_{rx}+sin\frac{j\pi x}{L}e^{-rt}V_x\right)^2}{\beta(y,z)V_{xx}}. \tag{3.1.5}$$

Inserting equation (3.1.5) back to equation (3.1.4) yields

$$\begin{aligned}&V_t+V_xxr+\tfrac{b(y)}{\xi}V_y+\delta C\left(z\right)V_z+\tfrac{1}{2\xi}a^2\left(y\right)V_{yy}+\tfrac{1}{2}g^2\left(z\right)\delta V_{zz}+\tfrac{\sqrt{\delta}}{\xi}a\left(y\right)g\left(z\right)P_{12}V_{yz}\\&+\tfrac{b}{\sqrt{\xi}}a\left(y\left(t\right)\right)V_{yr}+b\sqrt{\delta}g\left(z\left(t\right)\right)V_{Zr}+\tfrac{1}{2}b^2V_{rr}+V_r\left(\overline{a}\left(t\right)+b\tfrac{1}{L}\textstyle\sum_{j=1}^{N}x_k\left(j\right)W(j)e^{\frac{-\sigma}{2}^2j^2\pi^2t/L}\left(t\right)\right)\\&-\frac{1}{2}\frac{\left(\frac{a(y)P_1}{\sqrt{\xi}}V_{xy}+g(z)P_2\sqrt{\delta}V_{xz}+bV_{rx}+sin\frac{j\pi x}{L}e^{-rt}V_x\right)^2}{\beta(y,z)V_{xx}}=0,\end{aligned} \tag{3.1.6}$$

with terminal condition $v\left(T,x,y,z,r\right)=\left(xyz-\left(\lambda+w\right)\right)^2$.
We can verify that v has the form

$$v\left(t,x,y,z,r\right)=P\left(t,r\right)x^2y^2z^2-2\left(\lambda+w\right)Q\left(t,r\right)xyz+\left(\lambda+w\right)^2R(t,r)$$

with $P(t,r)>0$, $P(T,r)=1$, $Q(T,r)=1$, $R(T,r)=1$. Substituting the above expression into equation (3.1.6) we obtain the following partial differential equation for $P(t,r)$, $Q(t,r)$ and $R(t,r)$ respectively:

$$\begin{cases}P_t+2rP+\frac{b(y)P}{\xi}+\delta C\left(z\right)P+\frac{1}{2\xi}a^2\left(y\right)P+\frac{1}{2}g^2\delta P+\sqrt{\frac{\delta}{\xi}}a\left(y\right)g\left(z\right)P_{12}P+\frac{b}{\sqrt{\xi}}a\left(y\left(t\right)\right)P\\+b\sqrt{\delta}g\left(z\left(t\right)\right)P_r+\frac{1}{2}b^2P_{rr}+\left(\overline{a}+b\sin\frac{j\pi x}{L}e^{-rt}\right)P_r\\-\frac{\left[a(y)P_1P(t,r)+g(z)P_2\sqrt{\delta}P(t,r)+bP_r+\sin\frac{j\pi x}{L}e^{-rt}P\right]^2}{P}=0\\P\left(t,r\right)>0,P\left(T,r\right)=1\end{cases} \tag{3.1.7}$$

$$\begin{cases}Q_t+rQ+\frac{b(y)Q}{\xi}+\delta C\left(z\right)Q+\frac{1}{2\xi}a^2\left(y\right)Q+\frac{1}{2}g^2\delta Q+\sqrt{\frac{\delta}{\xi}}a\left(y\right)g\left(z\right)Q_{12}Q+\frac{b}{\sqrt{\xi}}a\left(y\left(t\right)\right)Q_r\\+b\sqrt{\delta}g\left(z\left(t\right)\right)Q_r+\frac{1}{2}b^2Q_{rr}+\left(\overline{a}+bsin\frac{j\pi x}{L}e^{-rt})\right)Q_r-\\\frac{\left(sin\frac{j\pi x}{L}e^{-rt}P+bP_r+g(z)P_2\sqrt{\delta}P+\frac{a(y)P_1P}{\xi}\right)\left(sin\frac{j\pi x}{L}e^{-rt}Q+bQ_r+g(z)Q_2\sqrt{\delta}+\frac{a(y)P_1Q}{\xi}\right)}{P}=0\\Q\left(t,r\right)=0,Q\left(T,r\right)=1\end{cases} \tag{3.1.8}$$

$$\begin{cases}R_t+\frac{b(y)}{\xi}a\left(y\left(t\right)\right)+\frac{b}{\sqrt{\xi}}a\left(y\left(t\right)\right)Q_r+b\sqrt{\delta}g\left(z\left(t\right)\right)R_r+\left(\overline{a}+bx_ksin\frac{j\pi x}{L}e^{-rt}\right)R_r\\+\frac{1}{2}b^2R_{rr}-\frac{\left(a(y)P_1Q+g(z)\sqrt{\delta}Q+bQ_r+sin\frac{j\pi x}{L}e^{-rt}Q\right)}{P}=0\\R\left(T,r\right)=1\end{cases}. \tag{3.1.9}$$

Let $P(t,r)$ *and* $Q(t,r)$ be of the form

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$$P\left(t,r\right)=e^{A(t)r+B(t)}, \qquad (3.1.10)$$

$$Q\left(t,r\right)=e^{C(t)+D(t)}, (3.1.11)$$

with terminal conditions

$$A\left(T\right)=B\left(T\right)=C\left(T\right)=D\left(T\right)=0 \qquad (3.1.12)$$

Deducingtheexpressionof$A(t)$,$B(t)$,$C(t)$*and*$D(t)$fromtheaboveequationinserting(3.1.10)and(3.1.7)gives$(A'(t)-\alpha A(t)+2)r(t)+$ $\beta'\left(t\right)-\frac{1}{2}b^2A^2+\left(\theta+b\sqrt{\delta}g\left(z\left(t\right)\right)+b\sin\frac{j\pi x}{L}e^{-rt}\right)A-\left(\frac{a(y)}{\xi}P_1\right)^2-\left(g(Z)P_2\sqrt{\delta}\right)^2-\left(\sin\frac{j\pi x}{L}e^{-rt}\right)^2=0.$

Substituting (3.1.11) into 3.1.8 results in $[C\left(t\right)-\alpha\left(t\right)+1]\,r\left(t\right)+2D'+2\delta C\left(z\right)+\frac{a^2(y)}{\xi}+g^2\left(z\right)\delta$

$$+2\sqrt{\frac{\delta}{\xi}}g\left(z\right)a\left(y\right)P_{12}+2\theta\left(t\right)C\left(t\right)+2b\sqrt{\delta}g\left(z\left(t\right)\right)C\left(t\right)+2bg\left(z\right)P_2\sqrt{\delta}C\left(t\right)$$
$$+b^2C^2\left(t\right)-2b^2C\left(t\right)A\left(t\right)-2bg\left(t\right)\sqrt{\delta}P_2A\left(t\right)-2b\frac{a(y)}{\sqrt{\xi}}P_1A\left(t\right)-\left[2bA\left(t\right)+sin\frac{j\pi x}{L}e^{-rt}\right]sin\frac{j\pi x}{L}e^{-rt}$$
$$+2\left(g\left(z\right)\sqrt{\delta}P_2\right)^2+2\left(\frac{a(y)}{\sqrt{\xi}}P_1\right)^2.$$

Bringing this two equations together with terminal condition (3.1.12) results in

$$A\left(t\right)=\frac{2}{\alpha}\left(1-e^{\alpha(T-t)}\right)=2C\left(t\right) \qquad (3.1.13)$$

$$B(t)=\int_{\boldsymbol{t}}^{\boldsymbol{T}}\left[-\frac{1}{2}\mathrm{b}^2\mathrm{A}^2+\theta(\mathrm{s})-\mathrm{b}\sqrt{\delta}\mathrm{g}(\mathrm{z})(\mathrm{s})+\mathrm{b}x_k\overline{W}e^{\frac{-\sigma}{2}^2\pi^2t}\mathrm{A}\right.$$
$$\left.-\left(\frac{\mathrm{a(y)}}{\xi}\mathrm{P}_1\right)^2-\left(\mathrm{g(z)P}_2\sqrt{\delta}\right)^2-sin\frac{j\pi x}{L}e^{-rt}\right] \qquad (3.1.14)$$

$$2D(t)=\int_t^T 2\theta(t)C(s)+2b\sqrt{\delta}g(z(t))C(t)$$
$$+2bg\left(z\right)P_2\sqrt{\delta}C\left(t\right)+b^2C^2\left(t\right)-2b^2C\left(t\right)A\left(t\right)-2bg\left(z\right)\sqrt{\delta}P_2A\left(s\right)$$
$$-2b\frac{a\left(y\right)}{\sqrt{\xi}}P_1A\left(t\right)-\left[2b-A\left(t\right)+\sin\frac{j\pi x}{L}e^{-rt}\right]\sin\frac{j\pi x}{L}e^{-rt}-2\left(g(z)\sqrt{\delta}P_2\right)^2-2\left(\frac{a(y)}{\sqrt{\xi}}P_1\right)^2$$
$$(3.1.15)$$

$$2D\left(t\right)-B\left(t\right)=-\int_t^T\left(\sin\frac{j\pi x}{L}e^{-rt}+bC(s)\right)^2ds+g\left(z\right)\sqrt{\delta}P_2+\left(\frac{a(y)}{\sqrt{\xi}}P_1\right)^2$$
$$(3.1.16)$$

By using the expressions of *P* (*t*, *r*) *and Q*(*t*, *r*) equation 9 becomes

$$R_t\left(t,r\right)+\frac{b}{\sqrt{\xi}}a\left(y\right)+b\sqrt{\delta}g\left(z\right)+\left(\overline{a}+b\sin\frac{j\pi x}{L}e^{-rt}\right)R_r\left(t,r\right)+\frac{1}{2}b^2R_{rr}\left(t,r\right)$$
$$-\left(\sin\frac{j\pi x}{L}e^{-rt}+bC\left(t\right)+\frac{a(y)}{\sqrt{\xi}}P_1+g(z)\sqrt{\delta}P_2\right)^2e^{2D(t)-B(t)}=0$$
$$(3.1.17)$$

So the solution of the above equation can be of the form:

$$R\left(t\right)=e^{2D(t)-B(t)}=e^{-\int_t^T\left(\psi(s)+bC(s)+\frac{a(y)}{\sqrt{\xi}}+g(z)\sqrt{\delta}\right)^2ds} \qquad (3.1.18)$$

The candidate for the optimal strategy has the form

$$\widehat{\pi} = \frac{(\lambda+w)\left(sin\frac{j\pi x}{L}e^{-rt}+bC(t)+\frac{a(y)}{\sqrt{\xi}}P_1+g(z)\sqrt{\delta}\right)}{\delta}e^{-C(t)+D(t)} - \frac{\left(sin\frac{j\pi x}{L}e^{-rt}+bA(t)+\frac{a(y)}{\sqrt{\xi}}PP_1+g(z)\sqrt{\delta}\right)}{\sigma(t)}x(t) \qquad (3.1.19)$$

And the respective candidate for the value function is of the form

$$V(t,x,y,z,r) = e^{A(t)r+B(t)}x^2y^2z^2 - 2(\lambda+w) + (\lambda+w)^2 e^{-\int_t^T \left(x_k sin\frac{j\pi x}{L}e^{-rt}+bC(s)+\frac{a(y)}{\sqrt{\xi}}+g(z)\sqrt{\delta}\right)^2 ds} \qquad (3.1.20)$$

With the expression of $A(t)$, $B(t)$, $C(t)$ and $D(t)$ as (3.1.13) – (3.1.15), we see the properties of $\widehat{\pi}(t, x)$ in [10].

Optimal Solutions and Efficient Frontier of Problem $P(w)$ According to the relationship between $PL1(\lambda,w)$ *and* $PL2(\lambda,w)$, if we define $\Gamma(\lambda)$ as the optimal objective function of $PL1(\lambda. w)$,then

$$\Gamma(\lambda) = P(0,r_0)x_0 - 2wQ(0,r_0)x_0 + w^2R(0) + \lambda^2(R(0)-1) + 2\lambda(wR(0)-Q(0,r_0)x_0)$$

Noting that $R(0) = e^{-\int_t^T \left(x_k sin\frac{j\pi x}{L}e^{-rt}+bC(s)+\frac{a(y)}{\sqrt{\xi}}+g(z)\sqrt{\delta}\right)^2 ds} < 1,$ then $\lambda^* = \arg(\max_{\lambda \epsilon R}\Gamma(\lambda))$ exists and is

$$\lambda^* = \frac{wR(0)-Q(0,r_0)x_0}{1-R(0)}$$

According to the relationship between $P(w)$ and $PL1(\lambda,w)$, the optimal strategy and efficient frontier of problem $P(w)$ are concluded in the theorem below.

Theorem 3.3. For problem $P(w)$, the optimal strategy is

$$\pi(t,x(t),y(t),z(t)) = \frac{w-Q(0,r_0)x_0,y_0,z_0}{1-R(0)}\frac{\left(bC(t)+x_k sin\frac{j\pi x}{L}e^{-rt}+\frac{a(y)}{\sqrt{\xi}}P_1+g(z)\sqrt{\delta}\right)}{\delta} - \frac{\left(bA(t)+x_k \sin\frac{j\pi x}{L}e^{-rt}+\frac{a(y)}{\sqrt{\xi}}PP_1+g(z)\sqrt{\delta}\right)}{\sigma(t)}x(t)$$

And the efficient frontier is

$$Var(X(T)) = \frac{R(0)}{1-R(0)}[EX(T)] - \frac{Q(0,r_0)}{R(0)}x_0$$

Where $R(0) = e^{-\int_t^T \left(x_k sin\frac{j\pi x}{L}e^{-rt}+bC(s)+\frac{a(y)}{\sqrt{\xi}}+g(z)\sqrt{\delta}\right)^2 ds}$.

3. Analysis of the Optimal Solution with Risk Premium

In this section, numerical simulations showing the relationship between the optimal control strategy and some sensitive parameters are presented. To achieve this, the following data are used similar to (21,30) unless otherwise stated: $w = 0.5$, $Q = 0.01$, $b = 0.01$, $C = 0.1$, $x_k = 0.5$, $x = 1$, $r = 0.1$, $T = 30$, $g = 0.1$, $a = 0.01$, $\xi = 0.01$, $P_1 = 0.01$, $P = 0.01$, $\delta = 0.01$, $\sigma = 0.1A = 20$

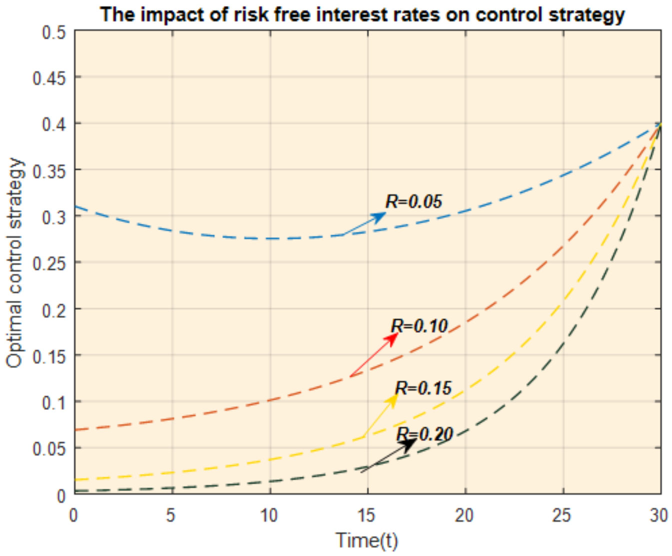

Figure 1: The Impact of Risk-Free Rates on Optimal Control Strategy

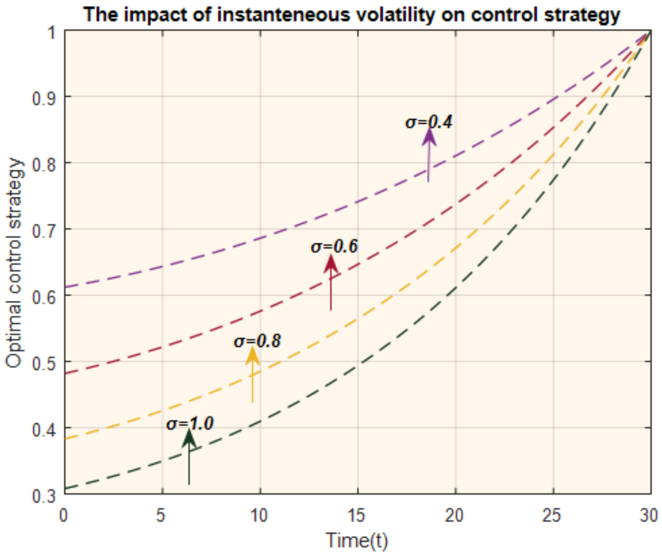

Figure 2: The Impact of Instantaneous Volatility on Optimal Control Strategy

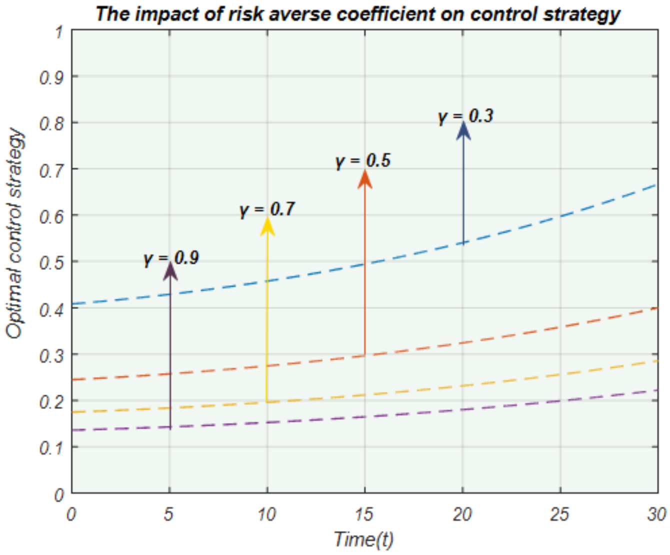

Figure 3: The Impact of Risk Premium Averse Coefficient On Optimal Control Strategy

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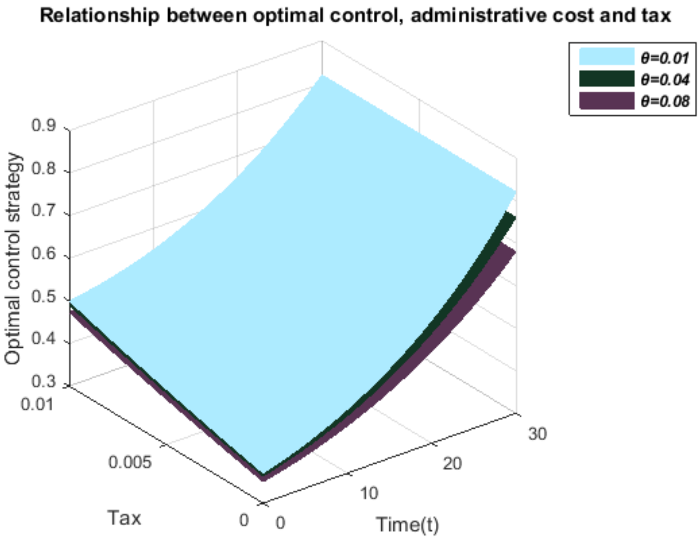

Figure 4: The Relationship Between Optimal Control Strategy, Administrative and Tax

The efficient frontier which gives a relationship between the expectation and the variance shows that the PPM's expectation is directly proportional to the variance; the implication of this is that, members who are willing to invest in highly risky assets have higher chances of getting more returns at the end of the investment period. i.e. more risk, implies higher expectation and vice versa. Also, from fig 1, the optimal control strategy decrease as the risk-free interest rate increases and increase when the risk-free interest rate decreases. This simply indicates that members will likely want to invest in risky asset when the interest rate from the risk-free asset is not attractive. However, if the risk-free interest rate is attractive enough, PPM members may be advised by their fund administrators to invest more in the risk-free asset, thereby reducing their investment in the risky asset. In figure 2, the graph of the optimal control strategy against the instantaneous volatility of the risky asset was presented; the plot shows that optimal control strategy is a decreasing function of the instantaneous volatility. Since the instantaneous volatility represents the risk premium of the risky asset, risk averse members with high instantaneous volatility will be scared of investing much in the risky asset, thereby investing more in the risk-free asset and vice versa. Figure 3, discuss the effect of the risk premium on the optimal control strategy and we observed that the optimal control strategy for the risky asset, is inversely proportional to the risk premium parameter. What we deduced from the graph in figure 3 is that members with higher risk premium may invest a lesser percentage of their wealth in the risky asset (stock) while members with lower risk premium may invest higher percentage of their wealth in the risky assets while reducing investment in the risk-free asset.

Figure 4, shows the relationship between the optimal control strategy, administrative and tax. We observed that the optimal control strategy of the PPM is a decreasing function of the administrative charges and an increasing function of the tax. The consequence of the graph in figure 4 is that if the administrative charges on investment of the risky asset is relatively low, the members may be encouraged to invest more in risky asset and may invest less if otherwise.

4. Conclusion

In this work, the optimal control strategy (OCS) for a PPM with return of contribution clause was investigated using multiscale stochastic volatility (MSSV) and mean variance model. The result of the study shows that the MSSV assumption is more realistic, though without some difficulties. It also shows that the introduction of two time-scales (a fast and a slow) volatility is efficient for capturing the main features of the observed term structures of implied (forecasted) volatility.

References

1. LeBaron, B. (2001). Stochastic volatility as asimple generator of apparent financial power laws and longmemory. Quantitative finance, 1(6), 621.
2. C. Olunkwa, B.O. Osu and Carlos Granados: Mean Variance portfolio selection problem with Multiscale Stochastic Volatility, *PROSPERTIVA 20,2,2022*.
3. Hillebrand, E. (2005). Neglecting parameter changes in GARCH models. *Journal of Econometrics, 129*(1-2), 121-138.
4. Finěk, V. (2016, December). Fractional step method for wavelet based solution of Black-Scholes equation. In *AIP Conference Proceedings* (Vol. 1789, No. 1, p. 030007). AIP Publishing LLC.
5. Hilber, N. (2009). Stabilized wavelet methods for option pricing in high dimensional stochastic volatility models (Doctoral dissertation, ETH Zurich).

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6. Hilber, N., Reichmann, O., Schwab, C., & Winter, C. (2013). Computational methods for quantitative finance: Finite element methods for derivative pricing. Springer Science & Business Media.
7. Fouque, J. P., Papanicolaou, G., Sircar, R., & Solna, K. (2003). Short time-scale in S&P500 volatility. Journal of Computational Finance.
8. Gatheral, J. (2011). The volatility surface: a practitioner's guide. John Wiley & Sons.
9. M. Chernov, R. Gallant, E. Ghysels, G. (2006). Tauchen: Alternative models for stock price *John Wiley and Sons.*
10. Musiela, M., & Rutkowski, M. (2006). Martingale methods in financial modelling (Vol. 36). Springer Science & Business Media.
11. Engle, R. F., & Patton, A. J. (2007). What good is a volatility model?. In Forecasting volatility in the financial markets (pp. 47-63). Butterworth-Heinemann.
12. Rometsch, M. (2010). A wavelet tour of option pricing (Doctoral dissertation, Ulm, Univ., Diss., 2010).
13. Brandt, M. W., & Diebold, F. X. (2003). A no-arbitrage approach to range-based estimation of return covariances and correlations.
14. Andersen, T. G., & Bollerslev, T. (1997). Intraday periodicity and volatility persistence in financial markets. Journal of empirical finance, 4(2-3), 115-158.
15. Osu, B. O., Eze, E. O., & Obi, C. N. (2020). The impact of stochastic volatility process on the values of assets. Scientific African, 9, e00513.
16. B. O. Osu, E.O. Eze, U.E. Obasi, H.I. (2020). Ukomah: Existence of solutions of some boundary value problems with stochastic volatility, Heliyon 6 (2) e03421. https://doi.org/10.1016/j. Heliyon..e03421. Published by Elsevier under the CC BY-NC-ND license.
17. Cooley, J., Lewis, P., & Welch, P. (2003). Application of the fast Fourier transform to computation of Fourier integrals, Fourier series, and convolution integrals. *IEEE Transactions on Audio and Electroacoustics, 15*(2), 79-84.
18. Atkinson, C., & Wilmott, P. (1995). Portfolio management with transaction costs: an asymptotic analysis of the Morton and Pliska model. Mathematical Finance, 5(4), 357-367.
19. Egere, A. C., Babalola, V. A., Bassey, U. N., & Egwe, M. E. (2025). Wavelets in Abstract Schwartz Spaces. Rocky Mountain Journal. To Appear.

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