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On Graded Deformations of the Universal Enveloping Algebra of a Color Lie Algebra

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Abstract
Let g be a Color Lie Algebra and $\mathcal{U}(\mathfrak{g})$ its the universal Enveloping Algebra. We define the notion of graded deformations and we give explicit graded deformations of the universal Enveloping Algebra of $\mathfrak{g}$.

Keywords: Algebra, Universal Enveloping, Graded Deformations

1. Premilinaries
Throughout this paper groups are assumed to be abelian and K is a field of characteristic zero. We recall some notation for graded algebras and graded modules and some facts on Color Lie algebras from [6-10].

1.1. Color Lie algebras
The concept of color Lie algebras is related to an abelian group G and an antisymmetric bicharacter $\varepsilon : G \times G \to \mathbb{K}^\times$, i.e.,

$$\varepsilon(g,h)\,\varepsilon(h,g) = 1, \tag{1.1}$$
$$\varepsilon(g,hk) = \varepsilon(g,h)\,\varepsilon(g,k), \tag{1.2}$$
$$\varepsilon(gh,k) = \varepsilon(g,k)\,\varepsilon(h,k), \tag{1.3}$$

where $g, h, k \in$ G and $\mathbb{K}^\times$ is the multiplicative group of the units in $\mathbb{K}$.

A G-graded space $L = \bigoplus_{g\in G} L_g$ is said to be a G-graded ε-Lie algebra (or simply, color Lie algebra), if it is endowed with a bilinear bracket [–,–] satisfying the following conditions

$$[L_g, L_h] \subseteq L_{gh}, \tag{1.4}$$
$$[a,b] = -\varepsilon(|a|,|b|)\,[b,a], \tag{1.5}$$
$$\varepsilon(|c|,|a|)\,[a,[b,c]] + \varepsilon(|a|,|b|)\,[b,[c,a]] + \varepsilon(|b|,|c|)\,[c,[a,b]] = 0, \tag{1.6}$$

where $g, h \in G$, and $a, b, c \in L$ are homogeneous elements.

For example, a super Lie algebra is exactly a $\mathbb{Z}_2$-graded ε-Lie algebra where

$$\varepsilon(i,j) = (-1)^{ij}, \quad \forall \quad i,j \in \mathbb{Z}_2. \tag{1.7}$$

1.2. Graded Cohomology of Cartan Eilenberg of Color Lie Algebra
A vector space M is G-graded if the is a family of subspaces $(M_g)g \in G$ such that $M = \bigoplus M_g$. Let M,N two graded vector spaces. A linear map from M to N is of homogeneous degree α if $f(M_\beta) \subset N_{\alpha+\beta}$ for β. Denote by $Hom_\alpha(M, N)$ the set the linear maps of homogeneous degree α. The graded vector space $Hom_{gr}(M,N) = \bigoplus Hom_\alpha(M,N)$ equipped by the bracket defined by $[f, g] = f \circ g - \varepsilon(f, g)\, g \circ f$ is a color Lie algebra. Let L be a color Lie algebra. A graded representation of L in M is a linear map of degree zero, $\rho : L \to Hom_{gr}(M, M)$ satisfying $[\rho(f), \rho(g)] = \rho([f, g])$. One said that M is a graded L module. In general a n linear map $f : L \times \cdots \times L \to M$ is of homogeneous degree α if $f(X\alpha_1, \cdots, X\alpha_n)$ is homogeneous of degree $\alpha + \alpha_1 + \cdots + \alpha_n$. Denote by $Hom^n_\alpha(L \times \cdots \times L, N)$ the set the linear maps of homogeneous degree α. The $Hom^n_{gr}(L \times \cdots \times L,N) = \bigoplus Hom^n_\alpha(L \times \cdots \times L,N)$ is a G graded vector space.

It admits a G graded L module given by $\rho(X)(f)(X_{\alpha_1},\cdots,X_{\alpha_n})=$

$$\rho(X)(f(X_{\alpha_1},\cdots,X_{\alpha_n}))-\sum_{i=1}^{n}\varepsilon(\alpha,\gamma+\alpha_1+\cdots\alpha_{i-1})f(X_{\alpha_1},\cdots,[X,X_{\alpha_r}],X_{\alpha_n}) \tag{1.8}$$

An element of f of homogeneous of degree γ is called ε skew symmetric if

$$f(X_{\alpha_1},\cdots,X_{\alpha_i},\cdots,X_{\alpha_j},X_{\alpha_n})=-\varepsilon(\alpha_i,\alpha_j)f(X_{\alpha_1},\cdots,X_{\alpha_j},\cdots,X_{\alpha_i},X_{\alpha_n}) \tag{1.9}$$

for $i<j$ We set
$C^0(L,N)=N$, $C^1_{gr}(L,N)=Hom_{gr}(L,N)$, $C^n_{gr}(L,N)$ the set of elements which are ε skew symmetric of $Hom^n_{gr}(L\times\cdots\times L,N)$. for $n>1$. The graded space called n graded cochains. The linear coboudary operator

$$d^n: C^n_{gr}(L,N)\to C^{n+1}_{gr}(L,N) \tag{1.10}$$

defined by for $n>1$

$$\delta^n(f)(x_1,\cdots,x_{n+1}) \tag{1.11}$$
$$=\sum_{i=1}^{n+1}(-1)^{i+1}\varepsilon(\gamma,\alpha_i)\varepsilon_i\, x_i\cdot f(x_1,\cdots,\hat{x_i},\cdots,x_{n+1}) \tag{1.12}$$
$$+\sum_{1\le i<j\le n+1}(-1)^{i+j}\varepsilon(\gamma,\alpha_i)\varepsilon(\gamma,\alpha_j)\varepsilon_i\varepsilon_j\, f([x_i,x_j],x_1,\cdots,\hat{x_i},\cdots,\hat{x_j},\cdots,x_{n+1}), \tag{1.13}$$

for all $f\in C^n_{gr}(L,N)_\gamma$, with $\varepsilon_i=\prod_{h=1}^{i-1}\varepsilon(|x_h|,|x_i|)$ $i\geq 2$, $\varepsilon_1=1$ and the sign indicates that the element below it must be omitted. We have $\delta^{n+1}\circ\delta^n=0$, [3, 9].

1.3. Graded Deformations of Color Lie Algebras

The field K admits a G graded by setting $K_e=K$ and $K_\gamma=0$ if $\gamma\neq 0$. Then the ring of series with coefficient in K, $K[[t]]$ is G graded by extension. Let E be a G-graded vector space. Then the $K[[t]]$ module $E[[t]]$ is G-graded module. Let L be a Color Lie algebra, The $K[[t]]$ module $L[[t]]$ is a Color Lie algebra by extension.

Definition 1 *Let L be a Color Lie algebra over K with multiplication ϕ_0.*
1. *A G graded deformation of the Color Lie algebra $(L[[t]],\phi_0)$ (of degree zero) is a G graded $K[[t]]$-module $L[[t]]$ equipped of a Color multiplication $\phi=\Sigma_t\phi_n t^n$ with $\phi_n\in C^2(L,L)_e$*
2. *Two graded deformations $(L[[t]],\phi_1)$ and $(L[[t]],\phi_2)$ of $L[[t]],\phi_0)$ are said equivalent if they are isomorphic i.e. there is a graded automorphism $f=id+tf_1+t^2f_2+\cdots+t^nf^n+\cdots$ with $f\in C^1(L,L)_e$ such that $\phi_2=f^{-1}\circ\phi_1\circ f\times f$.*
3. *A graded deformation $(L[[t]],\phi)$ is said trivial if all graded deformation is equivalent to $(L[[t]],\phi_0)$.*
4. *L is called graded rigid if all graded deformation of $(L[[t]],\phi_0)$ is trivial.*

Theorem 1 *Let L be a color Lie algebra. If the second graded cohomology $H^2(L,L)_e$ is equal to zero, then L is graded rigid.*

1.4 Graded Hochschild cohomology

Let G be an abelian group with identity element e. We will write G as an multiplicative group.

An associative algebra A with unit 1_A, is said to be G-graded, if there is a family $\{A_g|g\in G\}$ of subspaces of A such that $A=\bigoplus_{g\in G}A_g$ with $1_A\in A_e$ and $A_gA_h\subseteq A_{gh}$, for all $g,h\in G$. Any element $a\in A_g$ is called homogeneous of degree g, and we write $|a|=g$.

A (left) graded A-module M is a left A-module with an decomposition $M=\bigoplus_{g\in G}M_g$ such that $A_g.M_h\subseteq M_{gh}$. Let M and N be graded A-modules. Define

$$\mathrm{Hom}_{A\text{-gr}}(M,N)=\{f\in\mathrm{Hom}_A(M,N)|\ \ f(M_g)\subseteq N_g,\quad\forall\ \ g\in G\}. \tag{1.14}$$

Let us recall the notion of graded Hochschild cohomology of a graded algebra A. A graded A-bimodule is a A-bimodule $M=\bigoplus_{g\in G}M_g$ such that $A_g.M_h.A_k\subseteq M_{ghk}$. The Hochschild graded cochain complex of A is

$A \to Hom_{gr}(A, A) \to Hom_{gr}(A \times A, A) \to .. \to Hom_{gr}(A \times \cdots \times A, A) \to \cdots$ whose differential is given by

$$d(f)(a_0, a_1, \cdots, a_p) = a_0 f(a_1, \cdots, a_p) - \sum_{i=0}^{p-1} (-1)^i f(a_0, a_1, \cdots, a_i a_{i+1}, \cdots, a_p) + (-1)^p f(a_0, \cdots, a_{p-1}) a_p \tag{1.15}$$

1.5 Enveloping Algebra of a Color Lie algebra

Let L be a color Lie algebra as above and $T(L)$ the tensor algebra of the underlying G-graded vector space L. It is well-known that $T(L)$ has a natural $\mathbb{Z} \times G$grading which is fixed by the condition that the degree of a tensor $a_1 \otimes ... \otimes a_n$ with $a_i \in L_{gi}$, $g_i \in G$, $1 \le i \le n$, is equal to $(n, g_1 \cdots g_n)$. The subspace of $T(L)$ spanned by homogeneous tensors of order $\le n$ will be denoted by $T^n(L)$. Let $J(L)$ be the G-graded two-sided ideal of $T(L)$ which is generated by

$$a \otimes b - \varepsilon(|a|, |b|)\, b \otimes a - [a, b] \tag{1.16}$$

with homogeneous $a, b \in L$. The quotient algebra $U(L) := T(L) / J(L)$ is called the universal enveloping algebra of the color Lie algebra L. The $\mathbb{K}$-algebra $U(L)$ is a G-graded algebra and has a positive filtration by putting $U^n(L)$ equal to the canonical image of $T^n(L)$ in $U(L)$.

I (L) be the G-graded two-sided ideal of $T(L)$ which is generated by

$$a \otimes b - \varepsilon(|a|, |b|)\, b \otimes a \tag{1.17}$$

with homogeneous $a, b \in L$. The quotient algebra $S(L) := T(L) / I(L)$ is called the universal symetric algebra of the color Lie algebra L.

The canonical map $i : L \to U(L)$ is a G-graded homomorphism and satisfies

$$\mathtt{i}(a)\,\mathtt{i}(b) - \varepsilon(|a|, |b|)\,\mathtt{i}(b)\,\mathtt{i}(a) = \mathtt{i}([a, b]). \tag{1.18}$$

The $\mathbb{Z}$-graded algebra $G(L)$ associated with the filtered algebra $U(L)$ is defined by letting $G^n(L)$ be the vector space $U^n(L)/U^{n-1}(L)$ and $G(L)$ the space $\bigoplus_{n\in\mathbb{N}} G^n$ (L) (note $U^{-1}(L) := \{0\}$). Consequently, $G(L)$ is a $\mathbb{Z} \times G$-graded algebra. The well-known generalized Poincar´e-Birkhoff-Witt theorem, , states that the canonical homomorphism $i : L \to U(L)$ is an injective G-graded homomorphism; moreover, if $\{x_i\}_I$ is a homogeneous basis of L, where the index set I well-ordered. Set , then the set of ordered monomials $y_{k1} \bullet\bullet\bullet yk_n$ is a basis of $U(L)$, where $k_j \le k_{j+1}$ and $k_j < k_{j+1}$ if $\varepsilon(g_j, g_j) \neq 1$ with $x_{k_j} \in L_{g_j}$ for all $1 \le j \le n, n \in \mathbb{N}$. In case L is finite-dimensional $U(L)$ is a two-sided (graded) Noetherian algebra. The Poincare-Birkhoff-Witt theorem shows that there is a canonical graded algebra isomorphism from $S(L)$ to $G(L)$.

2. Graded Deformations of Enveloping algebra of A Color Lie Algebra

2.1. Graded Deformation by the Lie Color algebra

Theorem 2 *Let $\mathfrak{g}$ be a Color Lie Algebra and $\mathcal{U}(\mathfrak{g})$ its the universal Enveloping Algebra. If $(\mathfrak{g}[[t]],\mu_t)$ is a non trivial graded deformation of $\mathfrak{g}$, then it is uniquely extended to a graded deformation of the associative algebra $\mathcal{U}(\mathfrak{g})$.*

Proof: Let be a $(\mathfrak{g}[[t]], \mu_t)$ graded deformation of $\mathfrak{g}$ which is not non trivial with $\mu_t = \sum_{n=0}^{\infty} \mu_n t^n$ such that the class of μ_1 is not null in $\mathbf{H}^2(\mathfrak{g}, \mathfrak{g})_e$. Since $\mathfrak{g}$ is finite dimension, then the $\mathbb{K}[[t]]$-module $\mathfrak{g}[[t]]$ is graded isomorphic to the free module $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$. Let $e_1, \ldots, e_n$ be a homogeneous basis of $\mathfrak{g}$, $y_1, \ldots, y_n$ the vector of images of the basis in $\mathcal{U}(\mathfrak{g})$ and $y'_1, \ldots, y'_n$ the vector of images of the basis in $\mathcal{U}(\mathfrak{g}[[t]])$. Ones denotes • the multiplication of $\mathcal{U}(\mathfrak{g}[[t]])$ For all increase finite sequence $I = (i_1, \ldots, i_k)$ of indices in $\{1, \ldots, n\}$ set $y_I := y_{i_1} \cdots y_{i_k}$ in $\mathcal{U}\mathfrak{g}$ and $y'_I := y'_{i_1} \bullet \cdots \bullet y'_{i_k}$ in $\mathcal{U}(\mathfrak{g}[[t]])$. The theorem of Poincar´e-Birkhoff-Witt (is valid in the situation of Color Lie algebra which is a free module over a commutative ring, the y_I forme a basis of $\mathcal{U}(\mathfrak{g})$ over $\mathbb{K}$ and y'_I forme a basi of $\mathcal{U}(\mathfrak{g}[[t]])$ over $\mathbb{K}[[t]]$. We see the map $\Phi : Ug \otimes_{\mathbb{K}} \mathbb{K}[[t]] \to \mathcal{U}(\mathfrak{g}[[t]])$ defined by $\Phi(y_I) := y'_I$ is graded isomorphic of $\mathbb{K}[[t]]$ -modules. Let $\pi_t : \mathcal{U}\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]] \times Ug \otimes_{\mathbb{K}} \mathbb{K}[[t]] \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ the multiplication $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ induces by • et Φ, i.e., $\pi_t(a, b) := \Phi^{-1}(\Phi(a) \bullet \Phi(b))$. The restriction of π_t of elements of $\mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g})$ defined an application $\mathbb{K}$- bilinear $\mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{K}[[t]] \subset \mathcal{U}(\mathfrak{g})[[t]]$ that denote again by π_t, i.e, $\pi_t(u, v) = \sum_{n=0}^{\infty} t^n \pi_n(u, v)$ all elements $u, v \in \mathcal{U}(\mathfrak{g})$ ou $\pi_n \in Hom_{gr}(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g}))$. L'associativity of π_t on three elements u, v, w $\in \mathcal{U}(\mathfrak{g})$ imply the equations $\sum_{s=0}^{r} \left(\pi_s(\pi_{r-s}(u, v), w) - \pi_s(u, \pi_{r-s}(v, w)) \right) = 0$ all r $\in \mathbb{N}$. Then, πt defines a bilinear multiplication $\mathbb{K}[[t]]$-associative on the $\mathbb{K}[[t]]$-module $\mathcal{U}(\mathfrak{g})[[t]]$ (which is containing $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ as sub-module dense by the topology t-adic) of usually manner

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$$\pi_t\left(\sum_s^{\infty} t^s u_s, \sum_{s'=0}^{\infty} t^{s'} v_{s'}\right) := \sum_{r=0}^{\infty} t^r \sum_{\substack{s,s',s''\geq 0\\ s+s'+s''=r}} \pi_{s''}(u_s, v_{s'})$$

In particular, the map π_0 defines the associative multiplication on the vector space $\mathcal{U}(\mathfrak{g})$, and $(\mathcal{U}(\mathfrak{g})[[t]], \pi_t)$ is a graded associative deformation of $(\mathcal{U}(\mathfrak{g}), \pi_0)$. All finite increasing sequence *I, J* we have $\pi_0(y_I, y_J) = \Phi^{-1}(y'_I \bullet y'_J)|_{t=0}$: by 'well ordonning' the product $y'_I \bullet y'_J$ we obtain of linear combinaisons of y'_K ou the increasing finite sequence *K* is length less than or equal to the sum of the length of *I* and *J* which the coefficients do not contain only the structure constants of bracket μ_0 of the Color Lie algebra $\mathfrak{g}$ since $t = 0$. Then π_0 is equal the multiplication of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ et $(\mathcal{U}(\mathfrak{g})[[t]]$, is a graded associative deformation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$

It follows that π_1 is a graded 2-cocycle of Hochschild of $\mathcal{U}(\mathfrak{g})$, and the restriction of π_1 sur $X, Y \in \mathfrak{g}$ verifies

$$\mu_1(X,Y) = \pi_1(X,Y) - \varepsilon(X,Y)\pi_1(Y,X) \quad \forall X,Y \in \mathfrak{g}. \tag{2.1}$$

ce qui est ´equivalent `a

$$\sum_{r=0}^{\infty} t^r \sum_{\substack{a,b\geq 0\\ a+b=r}} \varphi_a(\pi_b(u,v)) = \sum_{r=0}^{\infty} t^r \sum_{\substack{a,b,c\geq 0\\ a+b+c=n}} \pi_a(\varphi_b(u), \varphi_c(v)) \qquad \forall u,v \in \mathcal{U}(\mathfrak{g}). \tag{2.2}$$

For $r = 1$, this relation gives

$$\pi_1(u,v) = (\delta_H\varphi_1)(u,v) \qquad \forall u,v \in \mathcal{U}(\mathfrak{g}) \tag{2.3}$$

where δ_H is the differential operator of Hochschild associated with the multiplication π_0 of the enveloping algebra.

Then the formulas gives

$$\mu_1(X,Y) = (\delta_H\varphi_1)(X,Y) - \varepsilon(X,Y)(\delta_H\varphi_1)(Y,X)$$

$$= X\varphi_1(Y) - \varphi(XY) + \varphi(X)Y - \varepsilon(X,Y)(-Y\varphi_1(X) + \varphi(YX) - \varphi(Y)X)$$

$$= (\delta_{CE}\varphi_1)(X,Y) \qquad \forall X,Y \in \mathfrak{g} \tag{2.4}$$

where δ_{CE} is the graded differential operator of Chevalley-Eilenberg
Then μ_1 est is coboundary of Graded Chevalley-Eilenberg and its class is null in

$\mathbf{H}^2(\mathfrak{g},\mathfrak{g})$e, gives a contradiction.

Corollary 1 *Let $\mathfrak{g}$ be a Color Lie Algebra and $\mathcal{U}(\mathfrak{g})$ its the universal Enveloping Algebra. If the associative algebra $\mathcal{U}(\mathfrak{g})$ is graded rigid, then the Color Lie algebra $\mathfrak{g}$ is graded rigid.*

2.2 Deformation by central extension

Let g be a Color Lie Algebra and ω a two graded cocycle with value in K such that its class in non null in $\mathbf{H}^2(\mathfrak{g}, K)_e$. The central extension of $\mathfrak{g}$ by c is defined by $\mathfrak{g}_\omega = \mathfrak{g} \oplus K$ and $[X + \alpha c, Y + \beta c] = [X, Y] + \omega(X, Y)_c$

Theorem 3 *Let g be a Color Lie Algebra and $\mathcal{U}_\mathfrak{g}$ its the universal Enveloping Algebra. If the second graded cohomology with in K, $H(\mathfrak{g}, K)_e$ is different from zero, then the Enveloping algebra $\mathcal{U}_\omega\mathfrak{g}$ admits a non trivial graded deformation.*

Proof: Let $\omega \in \mathbf{Z}^2_{CE}(\mathfrak{g},\mathbb{K})_e$ be a graded 2-cocycle of non null class and $\mathfrak{g}_{t\omega}[[t]]$ the one dimension central extension of the Color lie algebra $\mathfrak{g}[[t]] = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[[t]]$ over $K = \mathbb{K}[[t]]$ associated with tω . In the enveloping algebra $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$ of $\mathfrak{g}_{t\omega}[[t]]$ we denote the multiplication by • and we consider the bilateral ideal $\mathcal{I} := (1-c') \bullet \mathcal{U}(\mathfrak{g}_{t\omega}[[t]]) = \mathcal{U}(\mathfrak{g}_{t\omega}[[t]]) \bullet (1-c')$ (where c' is the image of c in $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$) and the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g} := \mathcal{U}(\mathfrak{g}_{t\omega}[[t]])/\mathcal{I}$. Let $e_1, \ldots, e_n$ a homogenous basis of $\mathfrak{g}$ over $\mathbb{K}$. Then $c, e_1, \ldots, e_n$ is a homogenous basis of $\mathfrak{g}_{t\omega}[[t]]$ sur $\mathbb{K}[[t]]$. Let $y_1, \ldots, y_n$ the images of basis vecteurs in $\mathcal{U}\mathfrak{g}$ et $c', y'_1, \ldots, y'_n$ the images of basis vecteurs in $\mathcal{U}(\mathfrak{g}_{t\omega}[[t]])$. For all finite increasing sequence $I = (i_1, \ldots, i_k)$ of indices in $\{1, \ldots, n\}$ set $y_I := y_{i_1} \cdots y_{i_k}$ in $\mathcal{U}\mathfrak{g}$ and $y'_I := y'_{i_1} \bullet \cdots \bullet y'_{i_k}$ in $\mathcal{U}(\mathfrak{g}[[t]])$. The theorem of Poincar´e-Birkhoff-Witt the y_I form a basis of $\mathcal{U}\mathfrak{g}$ over $\mathbb{K}$ and the $c'^{\bullet i_0} \bullet y'_I$ (où $i_0 \in \mathbb{N}$ et $c'^{\bullet i_0} := 1$) form a basis of $\mathcal{U}\mathfrak{g}_{t\omega}[[t]])$ over $\mathbb{K}[[t]]$. In the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}$, we see that $c'^{\bullet i_0}$ can be identified to 1. By denoting the multiplication in $\mathcal{U}_{t\omega}\mathfrak{g}$ by · and the images of $y'_1, \ldots, y'_n$ by the canonical projection by $y''_1, \ldots, y''_n$, wee for all y'_I is sending to $y''_I := y''_{i_1}\cdot\ldots\cdot y''_{i_n}$. It follows that all the elements y''_I forme a basis of the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}$. As in the demonstration of the pervious theorem , the map $\Phi : \mathcal{U}\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{K}[[t]] \to \mathcal{U}_{t\omega}\mathfrak{g}$ given by $y_I \mapsto y''_I$ defines an isomorphism of free $\mathbb{K}[[t]]$-modules. In the similarly manner as the precedent demonstration we show the multiplication induced over $\mathcal{U}\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{K}[[t]]$ by the multiplication · of $\mathcal{U}_{t\omega}\mathfrak{g}$ and Φ definies a suite of maps $\pi_t = \sum_{r=0}^{\infty}\pi_r t^r$, où $\pi_r \in \mathbf{Hom}(\mathcal{U}\mathfrak{g}\otimes\mathcal{U}\mathfrak{g},\mathcal{U}\mathfrak{g}_e$ with the following property: (1.) π_t defines a graded associative deformation $(\mathcal{U}\mathfrak{g}, \pi_0)$, i.e, a $\mathbb{K}[[t]]$-bilinear multiplication over the graded $\mathbb{K}[[t]]$-module $\mathcal{U}\mathfrak{g}\,[[t]]$ (wich containing the $\mathbb{K}[[t]]$-module $\mathcal{U}\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{K}[[t]]$ as sub-espace dense for the topology t-adic), et (2.) π_0 is usually multiplication of the enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$. It follows that π_1 is graded 2-cocycle of Hochschild of $\mathcal{U}\mathfrak{g}$, and over all $X, Y\in\mathfrak{g}\subset\mathcal{U}\mathfrak{g}$ we have the relation: $\omega(X, Y)1 = \pi_1(X, Y) - \varepsilon(X, Y)\pi_1(Y,X)$ since the Color Lie $\mathfrak{g}_{t\omega}[[t]]$ injecte in the quotient algebra $\mathcal{U}_{t\omega}\mathfrak{g}$, and so in $\mathcal{U}\mathfrak{g}\otimes_{\mathbb{K}}\mathbb{K}[[t]]\subset\mathcal{U}\mathfrak{g}\,[[t]]$.

We suppose that $\mathcal{U}\mathfrak{g}$ is rigid, then in particular, the deformation π_t is trivial. Then there exists a graded 1-cocycle of Hochschild $\varphi_1 \in C^1_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})$ such that $\pi_1 = \delta_H(\varphi_1)$. It follows that $\forall X, Y\in\mathfrak{g}$:

$$\omega(X,Y)1 = \pi_1(X,Y) - \varepsilon(X,Y)\pi_1(Y,X)$$

$$= \delta_H(\varphi_1)(X,Y) - \varepsilon(X,Y)\delta_H(\varphi_1)(Y,X) = \delta_{CE}(\varphi_1)(X,Y).$$

Then ω is a graded cobord of Chevalley-Eilenberg and its class is null in $\mathbf{H}^2_{CE}(\mathfrak{g},\mathbb{K})_e$, gives a contradiction.

2.3. Deformation by Poincaré-Birkhoff-Witt Theorem

Definition 2 *Let χ be bi-character of G. Let $A = \bigoplus A_g$ a graded χ commutative associative algebra. A Color Poisson bracket is a Color Lie bracket satisfying the Leibniz property $\{a\alpha, a\beta a\gamma\} = \{a_\alpha, a_\beta\}a_\gamma + \chi(\alpha, \beta)a_\beta\{a_\alpha, a_\gamma\}$. The algebra is called Color Poisson algebra.*

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Color Lie algebra $\mathfrak{g}$ equipped by the canonical filtration $(\mathcal{U}_n(\mathfrak{g}))_n$ satisfying $\mathcal{U}(\mathfrak{g})_n.\mathcal{U}(\mathfrak{g})_m\subset\mathcal{U}(\mathfrak{g})_{n+m}$. The graded associative algebra $gr(\mathcal{U}(\mathfrak{g})) = \oplus gr^n(\mathcal{U}(\mathfrak{g}))$ for this filtation, is defined by $gr^n(\mathcal{U}(\mathfrak{g})) = (\mathcal{U}(\mathfrak{g}))_n/(\mathcal{U}(\mathfrak{g}))_{n-1}$ with $(\mathcal{U}(\mathfrak{g}))_{-1} = 0$. Let $\pi_n : \mathcal{U}(\mathfrak{g}))_n \to gr^n(\mathcal{U}(\mathfrak{g}))$ the canonical projection. The multiplication of the algebra $gr(\mathcal{U}(\mathfrak{g}))$ is defined by

$$\pi_n(u)\pi_m(v) := \pi_{n+m}(u.v)$$

with $u\in\mathcal{U}(\mathfrak{g})_n$ and $v\in\mathcal{U}(\mathfrak{g})_m$

Theorem 4 *Let $(\mathfrak{g}, [,])$ be a Color Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. The associated graded algebra $gr(\mathcal{U}(\mathfrak{g}))$ is endowed of a Color Poisson bracket defined by*

$$\{\hat{u},\hat{v}\} := \pi_{n+m-1}(uv - \chi(u,v)vu)$$

with $u\in\mathcal{U}(\mathfrak{g})_n$ and $v\in\mathcal{U}(\mathfrak{g})_m$ and $\hat{u} = \pi_n(u),\ \hat{v} = \pi_m(v)$

Proof: The bracket $\{\hat{u},\hat{v}\}$ is independent of the choose of u and v. If $\hat{X},\hat{Y}\in gr^1(\mathcal{U}(\mathfrak{g}))$, then the bracket satisfies the properties of Poisson Color. This bracket extends on $gr(\mathcal{U}(\mathfrak{g}))$ by belinearity. The only property we need to verify is the Leibniz property . Let $\hat{X},\hat{Y},\hat{Z}\in gr^1(\mathcal{U}(\mathfrak{g}))$ and $X,Y,Z\in\mathcal{U}(\mathfrak{g})_1$ we have

$$\{\hat{X},\hat{Y}\hat{Z}\} = \pi_{1+2-1}(X(YZ) - \omega(X,YZ)(YZ)X)$$

$$= \pi_2((XY - \omega(X,Y)YX)Z + \omega(X,Y)Y(XZ - \omega(X,Z)ZX))$$
$$= ((\pi_1(X)\pi_1(Y) - \omega(X,Y)\pi_1(Y)\pi(X)\pi_1(Z) + \omega(X,Y)\pi_1(Y)(\pi_1(X)\pi(Z) - \omega(X,Z)\pi_1(Z)\pi_1(X))$$
$$= \{\hat{X},\hat{Y}\}\hat{Z} + \omega(\hat{X},\hat{Y})\{\hat{X},\hat{Z}\}$$

Since $\mathcal{U}(\mathfrak{g})_1 = \mathbb{K}\oplus\mathfrak{g}$, we have $\omega(\hat{X},\hat{Y}) = \omega(X,Y)$.

By Poincare-Birkhoff-Witt Theorem, this bracket regives the Color Lie algebra of $\mathfrak{g}$.

Definition 3 *Let $(A, m_0, \{,\})$ be a Color Poisson bracket. A graded star product of A is graded associative algebra deformation $(A[[t]], m_t)$ of A such that*

$$m_1(a_\alpha, a_\beta) - \chi(\alpha,\beta) m_1(a_\beta, a_\alpha) = \{a_\alpha, a_\beta\}$$

for $a_\alpha \in A_\alpha, a_\beta \in A_\beta$

Theorem 5 *Poincar´e-Birkhoff-Witt. Let* $(\mathfrak{g}, [,])$ *be a Color Lie algebra,* $\mathcal{U}(\mathfrak{g})$ *its universal enveloping algebra and the associated graded algebra* $gr(\mathcal{U}(\mathfrak{g}))$*. The symmetrization map* $\omega : gr(\mathcal{U}(\mathfrak{g})) \to \mathcal{U}(\mathfrak{g})$ *is homogenous isomorphism of degree zero of g module, defined by*

$$\omega_p(X_1 \bullet \cdots \bullet X_p) := \sum_{\sigma \in \mathcal{S}_p} \prod_{(i,j):\sigma(j)<\sigma(i)} \chi(i,j) X_{\sigma(1)} \circ \cdots \circ X_{\sigma(p)}$$

where $\mathbf{S}_p$ *is the group of permutation of ordr p and the product is extended over all r, s* ∈ 1, 2, ..., *n such that r < s and* $\pi^{-1}(r) > \pi^{-1}(s)$, • *is the multiplication of* $gr(\mathcal{U}(\mathfrak{g}))$ *and* ∘ *is the multiplication of* $\mathcal{U}(\mathfrak{g})$*. We have*

$$\mathcal{U}(\mathfrak{g}) = \oplus_{j \geqslant 0} \omega(gr^j(\mathcal{U}(\mathfrak{g}))$$

Theorem 6 *Let* $(\mathfrak{g}, [,])$ *be a Color Lie algebra,* $\mathcal{U}(\mathfrak{g})$ *its universal enveloping algebra and the associated graded algebra* $gr(\mathcal{U}(\mathfrak{g}))$*. The symmetrization map ω defines a star product over the Color Poisson algebra* $gr(\mathcal{U}(\mathfrak{g}))$ *given by*

$$u \star_t v = \sum_{n \geqslant 0} t^n \omega^{-1}((\omega(u) \circ \omega(v))_{p+q-n}) = \sum_{n=0}^{p+q-1} t^n \omega^{-1}((\omega(u) \circ \omega(v))_{p+q-n})$$

$where\ u \in gr^p(\mathcal{U}(\mathfrak{g})\ and\ v \in gr^q(\mathcal{U}(\mathfrak{g})\ .$

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