

Optimal Control Problem of Spacecraft Spatial Reorientation Taking into Account Loading of Construction Using Quaternions and The Combined Criterion of Quality

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Abstract

Quaternion solution of the problem on optimal control of a spatial turn of a spacecraft (as rigid body) from an arbitrary initial into an assigned final angular position taking into account the degree of loading of the construction is considered. The solved problem differs in use of new criteria of optimality. Optimization of control process is based on the combined functional of quality that combines in a given proportion time spent on spacecraft rotation and the integral of quadratic form relative to angular velocity (this quadratic form reflects dynamical loads on spacecraft construction). The proposed control method for spacecraft rotation improves conditions of turn in sense of minimum loading of spacecraft construction. Analytical solution of optimal control problem is obtained on the base of maximum principle with use of quaternionic models of the solid body motion controlled. The properties of optimal motion of a spacecraft are revealed in an explicit form, the structure of optimal control is specified. It was shown that degree of spacecraft construction loading during reorientation maneuver does not exceed the required value which is determined by coefficients of the minimized functional; and time of rotation is minimum also (as it is possible under given value of dynamical loading). To construct optimal control program, formalized equations and calculation formulas are written. Analytical equations and relations are presented for finding optimal control. Key relationships determining optimal values of the parameters of control algorithm for spacecraft turning are described. Numerical example and results of mathematical modeling that confirm the practical feasibility of the developed method for control of spacecraft reorientation are given. Important implication of the investigated problem of spacecraft optimal control is caused that often, allowable loading of spacecraft board and its elements of construction becomes significant restriction (constraints) for spacecraft motion with respect to center of mass (particularly for scientific spacecrafts and small spacecrafts which have sensitivity to loads). Solution to the problem of spatial reorientation in closed form is obtained for case of symmetric loading of spacecraft construction.

Keywords: Optimal Control, The Phase Variables, Quaternions, Attitude Control, The Tensor of Inertia, Control Function, The Maximum Principle, Conditions of Optimality, The Boundary-Value Problem

1. Introduction

The reorientation problem of a spacecraft (SC) from arbitrary initial angular position into the prescribed angular state was solved. Main difference of the proposed solution is new criteria of quality taking into account loading of spacecraft construction due to presence of rotation. Many papers have been devoted to the study of controlled rotations of a solid body around the center of mass and optimal control problems in various formulations; a wide range of solution methods was used [1-20]. A synthesis of optimal control based on analytical design of optimal regulators was described, or the spacecraft rotations constructed by concept of inverse problems of dynamics to obtain smooth controls providing the desired properties of the programmed trajectory to a class of polynomial of a given degree [1, 2]. Control of the spacecrafts equipped with inertial actuators has differential features [3-5]. The issues of optimal control are of special attention [1,5 20]. The reorientation problem for solids with varia mass-inertia configuration was considered. Many solutions apply the Pontryagin's maximum principle [6 20], but classical criterions of optimality of control (fast performance [5-9], minimum fuel consumption [6], minimum energy costs [10], etc.) have been used earlier. Kinematic problems of a turn have been studied in more detail [11-14], and the issues of control of spacecraft angular momentum are well known [12,14,20]. Particular solutions of the problems in dynamical formulation (as its analytical solutions) for optimal turns of spherically symmetric bodies [6,18] and

dynamically symmetric bodies are known also [7,14-17].

Below we solve the problem of optimal turn of solid body (SC) using new quality indicator taking account kinetical factor that is interpreted as degree of dynamical loads during spacecraft rotation. The combination of reorientation time and integral of kinetic factor leads to a limitation of spacecraft construction loadings (board and other elements) during rotation under maximal-fast maneuver. The considered problem differs from other problems of optimal control in the type of quality functional, which includes not only the control, but also duration of maneuver. Time factor limits a turnaround time.

2. Formulation of the Optimal Rotation Problem

The angular motion about the center of mass is described using the following equation [6]:

$$2\dot{\Lambda} = \Lambda \circ \omega \quad (1)$$

where ω is the vector of absolute angular velocity of a spacecraft; Λ is the normalized quaternion that specifies the motion of the spacecraft-associated coordinate system relative to the inertial coordinate system ($||\Lambda|| = 1$), « $\circ$ » is the sign of multiplication of quaternions [6,11-20].

The boundary conditions for spatial rotation controlled are:

$$\Lambda(0) = \Lambda_{in} \quad (2)$$

$$\Lambda(T) = \Lambda_f \quad (3)$$

where T is the time of maneuver completion.

The quaternions Λ_{in} and Λ_f satisfy the condition $||\Lambda_{in}|| = ||\Lambda_f|| = 1$. Note that the quaternions Λ and $-\Lambda$ correspond identical angular position, therefore we consider only the problems when $\Lambda_f \neq \pm\Lambda_{in}$. We assumed that optimum is the control which minimizes the following value

$$G = k_0 T + \int_0^T (k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2) dt \quad (4)$$

where ω_i are the projections of the absolute angular velocity vector ω onto the principal central axes of the spacecraft's ellipsoid of inertia (these axes form the associated); $k_0 > 0$ and $k_i > 0$ are constant positive coefficients ($i = \overline{1,3}$). Time of turn T is not fixed.

The optimal control problem is formulated in the following statement: it is required to transfer the spacecraft from attitude (2) to orientation (3) according to an equation (1) (with the boundary conditions for initial and final points of motion trajectory (2), (3)) so that (4) is minimal. Accepted criterion of quality (4) distinguishes the proposed optimization problem from the previously solved problems by the form of the minimized functional. Note that dynamic loadings can be estimated by the value $d = k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2$. The minimization of the index (4) limits the dynamic loadings during a turn (the coefficient $k_0 \neq 0$). Value of the coefficient k_0 determines maximal level of loading of construction elements caused by spacecraft rotation during turn maneuver (they arise due to a presence of non-zero angular velocity). Since duration T is not fixed, the requested reorientation maneuver can be performed under any conditions of the turn Λ_{in} and Λ_f and any values of k_1 , k_2 , k_3 and k_0 . Since the optimization is based on a combination of the quadratic quality criterion and the time of reorientation T (in a given proportion, with its own its own coefficients of proportionality), there is optimal time T_{opt} , relative to which the value (4) increases both with an increase and with a decrease in the time T . The problem of the optimal turn of a spacecraft with limited loads on board construction within minimum time period is very relevant.

3. Solving the Optimal Turn Problem

Let us find a solution to the formulation problem (1)–(4) on the base of the maximum principle of Pontryagin [21]. Since the minimized functional does not depend (explicitly) on the torque, the sought functions are angular velocities ω_i , and the desired control moment $\mathbf{M}$ is calculated with use of optimal angular velocity vector ω on the basis of the known [6]

$$I\dot{\omega} + \omega \times (I\omega) = \mathbf{M} \quad (5)$$

where $\mathbf{M}$ is control torque; I is the tensor of inertia of the spacecraft (the symbol $\times$ means the vector product of vectors).

The criterion of quality (4) not include elements of the attitude quaternion Λ , therefore, in place of the conjugate functions Ψ_j , corresponding to the components λ_j of quaternion Λ , we will use the following variables r_i ($i = \overline{1, 3}$) [18]:

$$r_1 = (\lambda_0 \psi_1 + \lambda_3 \psi_2 - \lambda_1 \psi_0 - \lambda_2 \psi_3) / 2, \quad r_2 = (\lambda_0 \psi_2 + \lambda_1 \psi_3 - \lambda_2 \psi_0 - \lambda_3 \psi_1) / 2, \quad r_3 = (\lambda_0 \psi_3 + \lambda_2 \psi_1 - \lambda_3 \psi_0 - \lambda_1 \psi_2) / 2$$

The optimal functions r_i and the vector $\mathbf{r}$ formed by r_i satisfy the equations

$$\dot{\mathbf{r}} = -\boldsymbol{\omega} \times \mathbf{r}, \quad \dot{r}_1 = \omega_3 r_2 - \omega_2 r_3, \quad \dot{r}_2 = \omega_1 r_3 - \omega_3 r_1, \quad \dot{r}_3 = \omega_2 r_1 - \omega_1 r_2 \quad (6)$$

The Hamilton-Pontryagin function is

$$H = \omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3 - k_0 - k_1 \omega_1^2 - k_2 \omega_2^2 - k_3 \omega_3^2$$

When we write the Hamilton-Pontryagin function, the constraint $||\Lambda|| = 1$ was not taken into account due to the equality $||\Lambda(0)|| = 1$, which was mentioned earlier. The vector $\mathbf{r}$ is stationary relative to the inertial basis, and $|\mathbf{r}| = \text{const} \neq 0$ (the constancy of the modulus $|\mathbf{r}|$ follows from properties of the equations (6)). The specific solution $\mathbf{r}(t)$ of system (6) is determined by the initial Λ_{in} and final Λ_f positions of the spacecraft. The optimal function $\mathbf{r}(t)$ is calculated using the quaternion $\tilde{\Lambda}(t)$ [6, 18]:

$$\mathbf{r} = \tilde{\Lambda} \circ \mathbf{c}_E \circ \Lambda, \quad \text{where } \mathbf{c}_E = \text{const} = \Lambda_{in} \circ \mathbf{r}(0) \circ \tilde{\Lambda}_{in}$$

where $\mathbf{r}(0) \neq 0$ (otherwise $r_1 = r_2 = r_3 \equiv 0$ and there is no point in solving the problem further). Here $\tilde{\Lambda}$ is a quaternion conjugate to the quaternion Λ [6,10-22].

The problem on finding optimal control consists of solving the system of equations (1), (5), (6) with constraint $H(T)=0$ with simultaneous maximization of the function H at each current moment of time t and fulfillment of the boundary conditions (2), (3). We introduce $r_0 = |\mathbf{r}(t)| = \text{const} \neq 0$ and the normalized vector $\mathbf{p} = \mathbf{r} / |\mathbf{r}|$, $p_i = r_i / r_0$. For the vector $\mathbf{p}$ and its components p_i , the equations

$$\dot{p}_1 = \omega_3 p_2 - \omega_2 p_3, \quad \dot{p}_2 = \omega_1 p_3 - \omega_3 p_1, \quad \dot{p}_3 = \omega_2 p_1 - \omega_1 p_2, \quad \dot{\mathbf{p}} = -\boldsymbol{\omega} \times \mathbf{p} \quad (7)$$

are satisfied. The Hamiltonian H is a quadratic function of the angular velocities ω_i , and its maximum is determined by the necessary conditions for the extremum $\partial H / \partial \omega_i = 0$, from which we find the optimal angular velocities

$$\omega_i = r_0 p_i / (2k_i) \quad (8)$$

or

$$\omega_i = b(t) p_i / k_i \quad (9)$$

The solution $\boldsymbol{\omega}(t)$ of the system of differential equations

$$\dot{\omega}_1 = \omega_2 \omega_3 (k_2 - k_3) / k_1, \quad \dot{\omega}_2 = \omega_1 \omega_3 (k_3 - k_1) / k_2, \quad \dot{\omega}_3 = \omega_1 \omega_2 (k_1 - k_2) / k_3 \quad (10)$$

is optimum. The functions $\omega_i(t)$, satisfying the system (10) and (1) with simultaneous fulfillment of the conditions (2), (3) are the desired angular velocities. Accordingly, optimal trajectory $\Lambda(t)$ is the solution of equations system (1), (2), (3), (10). The value $d = k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2$ can characterize a degree of spacecraft loading (by a choice of k_1, k_2, k_3), it is of importance and it requires attention. The acceleration at a certain point of a spacecraft is calculated using an expression

$$\mathbf{a} = \mathbf{a}_0 + \boldsymbol{\zeta} \times \mathbf{R} + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times \mathbf{R}]$$

where $\mathbf{R}$ is the radius-vector of the controlled point of a spacecraft relative to center of mass; $\mathbf{a}_0$ is an acceleration of mass center with respect to inertial coordinate system; $\boldsymbol{\zeta} = \dot{\boldsymbol{\omega}}$ is angular acceleration of a spacecraft.

The point $\mathbf{R}$ is not move relative to spacecraft coordinate system $OXYZ$ ($\mathbf{R} = \text{const}$), linear acceleration is absent (with respect to $OXYZ$), and the Coriolis acceleration is a zero vector. For orbital $\mathbf{a}_0 \approx 0$, and $\boldsymbol{\zeta}$ is small (or momentary). Last term (last

vector product) reflects the spacecrafts loads that is very interesting. As a rule, the degree of loading of the spacecraft body is determined using several key characteristic points (as minimum, along spacecraft axe OX, OY, OZ). We suppose that η_1, η_2, η_3 are coordinates of critical points $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$; a_1, a_2, a_3 are the weight coefficients in the direction of spacecraft axes OX, OY, OZ , respectively ($a_1 > 0, a_2 > 0, a_3 > 0$). The loading of a spacecraft construction due to rotation around center of mass is estimated by the following value

$$\mu = a_1 |\eta_1| (\omega_2^2 + \omega_3^2) + a_2 |\eta_2| (\omega_1^2 + \omega_3^2) + a_3 |\eta_3| (\omega_1^2 + \omega_2^2)$$

(it is summary loads as general estimation). The coefficients k_i connect with coordinates η_i and ве-сами a_i by the dependences:

$$k_1 = a_2 |\eta_2| + a_3 |\eta_3|, \quad k_2 = a_1 |\eta_1| + a_3 |\eta_3|, \quad k_3 = a_1 |\eta_1| + a_2 |\eta_2|$$

Thus, the loads caused by the existence of an acceleration, arising during spacecraft rotation, are estimated by the value d (if $\mathbf{R}_1 = \{\eta_1, 0, 0\}, \mathbf{R}_2 = \{0, \eta_2, 0\}, \mathbf{R}_3 = \{0, 0, \eta_3\}$).

The motions corresponding to the equations (7), (9) satisfy the equality

$$p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3 = \text{const} \quad (11)$$

For proof of (11), we differentiate the left-hand side of (11) and verify that the resulting derivative is zero after substituting $\dot{\mathbf{p}}_i$ into the system (7), and then ω_i according to relations (9). Consequently, the dynamical loading d is constant during optimal rotation of a spacecraft (since $|\mathbf{r}|=\text{const}$).

The problem on constructing the optimal program of spacecraft turn was reduced to solving the system of equations of motion (1) and the equations (7) together with the equalities $r_i = r_0 p_i$ by virtue of the conditions (8) for the angular velocities ω_i . Time of reorientation finish T is not fixed, and the Hamiltonian H does not explicitly depend on time. Therefore, at each instant $t \in [0, T]$ optimal control must satisfy the condition $H = \text{const} = 0$ [22]. From condition of maximum for the function H , and concretely $r_i = 2k_i \omega_i$, the value of the Hamiltonian is calculated

$$H = k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2 - k_0 = 0$$

and obtained the necessary condition of optimality: $k_0 = k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2$. We come to a conclusion that spacecraft rotates with the loads $d = k_0$ during optimal turn. Taking into account (8), we write

$$r_0^2 (p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3) = 4k_0$$

The motions which correspond to the equations (7), (8) have the feature (11) within the entire interval of time. The constant r_0 is related with k_0 as $r_0 = 2\sqrt{k_0}/C$, where $C = \sqrt{p_{10}^2/k_1 + p_{20}^2/k_2 + p_{30}^2/k_3}$; p_{10}, p_{20}, p_{30} are the components of vector $\mathbf{p}_0 = \mathbf{p}(0)$. Optimal angular velocities ω_i are

$$\omega_i = \sqrt{k_0} p_i / (k_i C) \quad (12)$$

For forming the programmed angular velocities ω_i , it is necessary to know the behavior of the function p_1, p_2, p_3 . Optimum is the solution $\mathbf{p}(t)$ of the following system of differential equations

$$\dot{p}_1 = \frac{\sqrt{k_0}(1/k_3 - 1/k_2)p_2 p_3}{\sqrt{p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3}}; \quad \dot{p}_2 = \frac{\sqrt{k_0}(1/k_1 - 1/k_3)p_1 p_3}{\sqrt{p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3}}; \quad \dot{p}_3 = \frac{\sqrt{k_0}(1/k_2 - 1/k_1)p_1 p_2}{\sqrt{p_1^2/k_1 + p_2^2/k_2 + p_3^2/k_3}} \quad (13)$$

The functions $\omega_i(t)$, satisfying the system (12), (13) and (1) with simultaneous satisfaction of conditions (2), (3) are the sought optimal angular velocities. Accordingly, optimal trajectory is the solution of the equation system (1), (2), (3), (12), (13). The problem of constructing the optimum control consists of finding such $\mathbf{p}_0$, for which the solution of the system (1), (12), (13) with initial condition $\Lambda(0) = \Lambda_m$ satisfies the equality $\Lambda(T) = \Lambda_f$. Key difficulty is in determining such boundary vectors $\mathbf{p}(0), \mathbf{p}(T)$, for which

$$\Lambda_f \circ \mathbf{p}(T) \circ \tilde{\Lambda}_f = \Lambda_{in} \circ \mathbf{p}(0) \circ \tilde{\Lambda}_{in} \text{ or } \mathbf{p}(T) = \tilde{\Lambda}_f \circ \mathbf{p}_0 \circ \Lambda_f$$

where $\Lambda_f = \tilde{\Lambda}_{in} \circ \Lambda_f$ is the quaternion of a turn.

It is required to notice that use of combined criteria of optimality (4) guarantees a rotation of a spacecraft with loading of construction does not exceed preset level, and final time T is minimum.

4. Designing Optimal Control Program of Rotation

In practical problems, as a rule, in initial and final instants spacecraft's angular velocity ω is zero. Therefore, transition segments must be (spin-up of a spacecraft to design angular velocity at beginning of a turn and braking, i.e. cancellation of existing velocity to zero at the end of turn maneuver). We denote $f = \sqrt{k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2}$ and $f_0 = \max_{0 \leq t \leq T} f(t)$ (for simplification in the future). If durations of spin-up Δt_{ac} and braking Δt_{br} is small (in comparison with the time of a turn T), then на участках разгона и торможения the derivative $\dot{f}$ is practically constant $\dot{f} \approx \text{const} \neq 0$, change in value $f(t)$ can assume linear (the value d is close to a quadratic function of time). It is supposed that $f^2(t) = u_{ac} t^2$ during spin-up, and $f^2(t) = u_{br} (T-t)^2$ when braking (u_{ac} and u_{br} can be identical or different, since the times Δt_{ac} , Δt_{br} can be equal, but can be $\Delta t_{ac} \neq \Delta t_{br}$ therefore $u_{ac} = f_0^2 / \Delta t_{ac}^2$, $u_{br} = f_0^2 / \Delta t_{br}^2$). Important characteristic of spatial rotation is the path functional

$$S = \int_0^T \sqrt{k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2} dt$$

We obtain $S = f_0(T - (\Delta t_{ac} + \Delta t_{br})/2)$ under the assumptions made above. The indicator (4) is estimated by the following sum

$$G = f_0^2 (T - 2(\Delta t_{ac} + \Delta t_{br})/3) + k_0 T \quad (14)$$

For optimal turn, the time $T \approx S/f_0 + (\Delta t_{ac} + \Delta t_{br})/2$. Substitute this value T into (14) we have

$$G = f_0^2 (S/f_0 - (\Delta t_{ac} + \Delta t_{br})/6) + k_0 (S/f_0 + (\Delta t_{ac} + \Delta t_{br})/2)$$

We see that if $\Delta t_{ac} + \Delta t_{br} = 0$ then the index (4) achieves minimum when $f_0 = \sqrt{k_0}$. For the considered model of optimal turn, in sense of minimum of (4), the sum $\Delta t_{ac} + \Delta t_{br}$ is proportional to the value f_0 , i.e. $\Delta t_{ac} + \Delta t_{br} = f_0 h$, where $h = \text{const} > 0$ is constant positive coefficient of proportionality. Taking into account a presence of acceleration and braking, $G = S(f_0 + k_0/f_0) + (\Delta t_{ac} + \Delta t_{br})(k_0/2 - f_0^2/6)$. Whence $G = k_0 S/f_0 + (S + k_0 h/2)f_0 - h f_0^3/6$. extremum point is determined using the equation $\partial G / \partial f_0 = 0$ or $S + h k_0/2 - k_0 S / f_0^2 - h f_0^2/2 = 0$, the solution of which is $f_0 = \sqrt{k_0}$. Hence, dynamical loading of construction does not exceed a priori given threshold k_0 during spacecraft rotation. It is known that the path functional S is minimal for optimal motion in accordance with criterion (4) [13]. Therefore, the time of turn is minimum.

Thus, we have demonstrated that minimization of the functional (4) minimizes the turning time T and value of dynamic loads acting on elements of spacecraft construction during its rotation from initial position Λ_{in} in a given final position Λ_f does not exceed certain preset value. At impulse control, when Δt_{ac} , Δt_{br} are negligibly small ($\Delta t_{ac} \rightarrow 0$, $\Delta t_{br} \rightarrow 0$), the solution (7), (8) is absolute optimality.

If $|\mathbf{M}| \leq m_0$, then for short transition stages Δt_{ac} , Δt_{br} are estimated by the value

$$\Delta t_{br} \approx \Delta t_{ac} = \tau \approx f_0 \sqrt{J_1^2 p_{10}^2 / k_1^2 + J_2^2 p_{20}^2 / k_2^2 + J_3^2 p_{30}^2 / k_3^2} / (m_0 C)$$

where m_0 is the maximal magnitude of control torque $\mathbf{M}$, J_1, J_2, J_3 are the principal central moments of inertia of a spacecraft.

Control functions are formed according to (8), for which we must know p_1, p_2, p_3 at each instant of time. The control torque is calculated using (5). In general case, analytical solution of the system (1), (7), (9) was not found (for an arbitrary load distribution). However, we will note two special (specific) cases. For symmetric variant of spacecraft loading (when $k_1 = k_2 = k_3$), the dependences are a sufficient trivial:

$$p_i(t) = \text{const} = p_{i0} = v_i / \sqrt{v_1^2 + v_2^2 + v_3^2}, \quad \omega_i(t) = b(t) p_{i0} / k_i$$

where v_0, v_1, v_2, v_3 are the elements of the quaternion of a turn $\Lambda_t = \Lambda_{in} \circ \Lambda_f$; and time function $b(t)$ is specified by the coefficient k_0 . We write the trajectory $\Lambda(t)$ in analytical form

$$\Lambda(t) = \Lambda_{in} \circ e^{\mathbf{p}_0 s(t)/(2k_1)}, \quad s(t) = \int_0^t b(t) dt$$

For axial-symmetric loading of a spacecraft ($k_2 = k_3$), the problem of optimal control can be reduced to analytic solution (the axis of symmetry is considered to be the OX axis). Optimal motion is simultaneous rotation of a spacecraft around the direction specified by the vector $\mathbf{p}$, stationary relative to the inertial basis, and around the axis OX , forming constant angle ϑ with $\mathbf{p}$ (from (7) we have $p_1 = \text{const} = p_{10} = \cos \vartheta$, $\vartheta = \arccos p_{10}$). Angular velocity about $\mathbf{p}$ and OX change proportionally with a constant coefficient of proportionality. The boundary-value problem of a turn in this special case (particular ratio between the coefficients k_i) is reduced to solving the system of two equations

$$\Lambda_f = \Lambda_{in} \circ e^{\mathbf{p}_0 \beta/2} \circ e^{\mathbf{e}_1 \alpha/2}, \quad \text{and} \quad \alpha = \frac{J - k_1}{k_1} p_{10} \beta \quad (15)$$

that provides directly calculate the required vector $\mathbf{p}_0$. Optimal functions p_i correspond to the law:

$$p_1 = p_{10} = \cos \vartheta, \quad p_2 = p_{20} \cos \kappa + p_{30} \sin \kappa, \quad p_3 = -p_{20} \sin \kappa + p_{30} \cos \kappa, \quad \kappa = \frac{J - k_1}{J} \int_0^t \omega_1(t) dt$$

where $p_{i0} = p_i(0)$; $J = k_2 = k_3$; longitudinal velocity $\omega_1(t)$ is determined using (12) taking account при том, что $p_1 = \text{const} = p_{10}$. Значения α , β и p_{i0} рассчитываются по значениям Λ_{in} and Λ_f из системы (15).

In the case of an arbitrary distribution of the loads ($k_1 \neq k_2 \neq k_3$) the solution of the system (1), (7), (9) is found only by numerical methods (for example, by the method of successive approximations [23,24]). The desired vector $\mathbf{p}_0$ is obtained from the solution of the boundary-value problem $\mathbf{p}(0) = \mathbf{p}_0$, $\mathbf{p}(T) = \tilde{\Lambda}_t \circ \mathbf{p}_0 \circ \Lambda_t$ for system (7), (9). Using the known property of independence of optimal vector $\mathbf{p}_0$ from the behavior of the function $b(t)$ [13], we compute $\mathbf{p}_0$ under the assumption $b(t) = \text{const}$, that significantly simplifies the calculation of $\mathbf{p}_0$.

It is necessary to note that the larger k_0 in the performance index (4), the larger the maximum loading of spacecraft construction (and the value f_0 , accordingly). On the other hand, a decrease in the coefficient k_0 leads to an increase in the duration of reorientation T .

4. Example of Solving the Problem of Optimal Turn and Mathematic Simulation

Demonstrating the behavior of motion parameters in visual form is considered on example of 180° turn of some spacecraft into position corresponding to a quaternion Λ_f with the following elements: $\lambda_0 = 0$; $\lambda_1 = 0.7071$; $\lambda_2 = 0.5$; $\lambda_3 = 0.5$. The directions of same axes of the bound and inertial coordinate systems coincide in initial position, and $\omega(0) = \omega(T) = 0$ also. Below we give numeric decision of the controlled reorientation that is optimal in sense of a minimum of sum (4), where $k_0 = 0.02533 \text{ s}^{-2}$, and $k_1 = 60$, $k_2 = 187$, $k_3 = 160$. Inertial characteristics of a spacecraft are accepted: $J_1 = 26516 \text{ kg}\cdot\text{m}^2$, $J_2 = 102045 \text{ kg}\cdot\text{m}^2$, $J_3 = 96612 \text{ kg}\cdot\text{m}^2$. Maximal control torque is $m_0 = 160 \text{ N}\cdot\text{m}$.

According to the maximum principle the necessary conditions of optimality are the equations (6), condition of Hamiltonian maximum and the requirement $H = 0$, which should be satisfied at each current instant $t \in [0, T]$. The velocities (12) satisfy all three conditions, and the equations (13) are valid for the system (6) taking the relations $r_i = r_0 p_i$ into account. Solving the two-point boundary-value problem of a turn $\mathbf{p}(0) = \mathbf{p}_0$, $\mathbf{p}(T) = \tilde{\Lambda}_t \circ \mathbf{p}_0 \circ \Lambda_t$ for the system (13) give $\mathbf{p}_0 = \{0.4822264; -0.116449; 0.8682726\}$. The constant r_0 is $r_0 = 3.43 \text{ s}^{-2}$. The turnaround time is $T_{\text{opt}} = 200.2 \text{ s}$. End of spin-up t_1 and beginning of braking t_2 are $t_1 = 6.1 \text{ s}$ and $t_2 = 194.4 \text{ s}$. Maximal loading (between acceleration and braking) is $d_{\text{max}} = f_0^2 = 0.02533 \text{ s}^{-2}$, and maximal modulus of angular momentum $L_{\text{max}} = 968 \text{ Н}\cdot\text{м}\cdot\text{с}$. Note that optimum is $d = f_0^2 = k_0$ for ideal reorientation (for kinematical problem of a turn).

A demonstration of spacecraft motion during optimal turn are illustrated on figures 1–3 (using the mathematical modeling data). The figure 1 shows a character of behavior of projections $p_1(t)$, $p_2(t)$, $p_3(t)$. The graphs of angular velocities (in

spacecraft coordinate system) $\omega_1(t)$, $\omega_2(t)$, $\omega_3(t)$, as functions of time, are demonstrated on figure 2 (ω_i given in degrees/s). Figure 3 illustrates the change in components of attitude quaternion $\Lambda(t)$ for the turn maneuver performed ($\lambda_0(t)$, $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ reflect spatial orientation in time). The variables p_i , λ_j are dimensionless quantities. For optimal control, ω_i are the continuous functions of time, and p_i and λ_j are the smooth functions of time. Note that ω_1 and p_1 are of constant sign, and p_1 has almost constant value between spin-up and braking (in contrast to p_2 and p_3), we see the insignificant variation of the projection ω_1 . It means that OX is the longitudinal axis of a spacecraft.

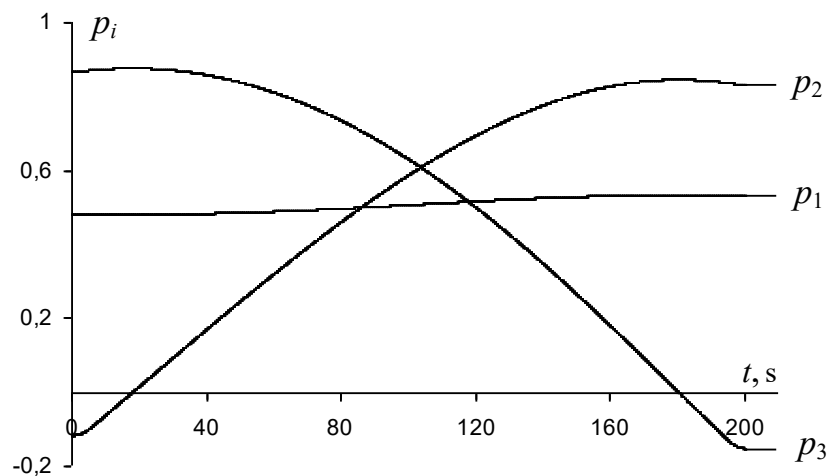


Figure 1: The Functions $p_1(t)$, $p_2(t)$, $p_3(t)$ During Spatial Reorientation

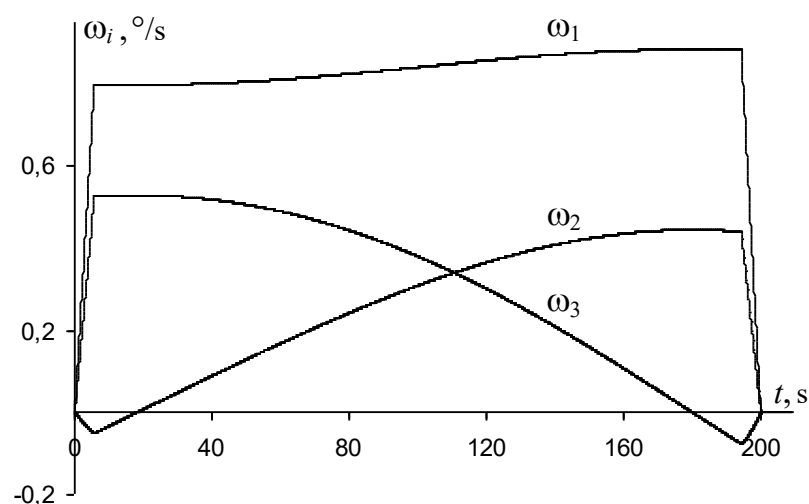


Figure 2: Change in Angular Velocities of a Spacecraft During an Optimal Turn

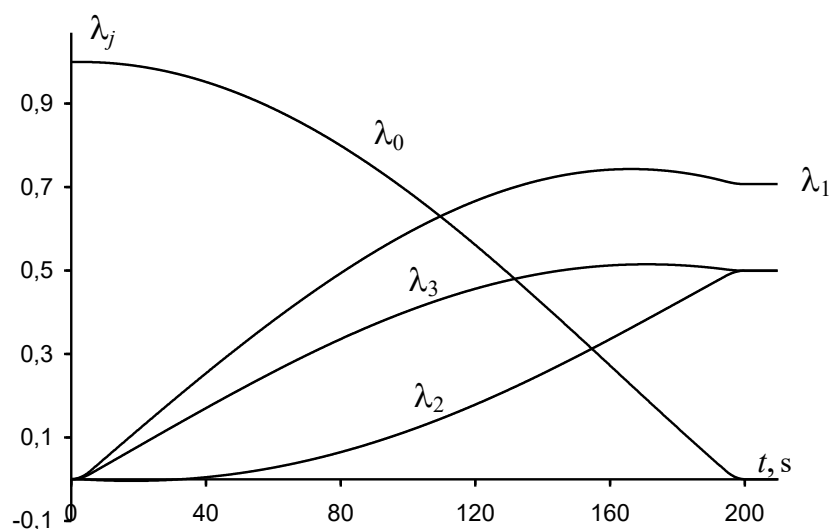


Figure 3: Change in the Elements of Orientation $\Lambda(t)$ During an Optimal Turn

5. Conclusion

The problem of optimal control for spacecraft rotation from initial angular position into given finish orientation is examined. Main difference of the presented solution is the optimization on the base of new functional of quality combined in given proportion time of rotation and the integral of quadratic form of angular velocity during a turn. The sub-integrative expression in the functional (4) reflects a degree of dynamical loads on spacecraft construction. The obtained control of spacecraft rotation improves conditions of a turn in sense of minimization of loading of spacecraft construction. Since in many cases the spacecraft motion around center of mass is limited by maximum permissible loading of spacecraft board and other elements of construction, the problem investigated by us is of practical importance (some scientific spacecrafts have sensitivity to dynamical loads). To solve the formulated control problem of spacecraft taking into account loading of body construction, quaternion models and the maximum principle of Pontryagin are used. The Hamilton–Pontryagin function, the adjoint system of equations and analytic relations and formulas for optimal control function are written for the proposed problem of optimization. The optimum control structure based on the necessary conditions of optimality is specified. We describe in detail key properties of a turn and the trajectory type satisfying to the adopted criteria (4). The relationships for definition of spatial rotation of a spacecraft are presented. The implementation of a program turn was described. It was noted that the sub-integrative expression in (4) can be correspond to degree of loading of spacecraft construction (due to dynamic loads arising during rotation). If take into account the told above, can understand a choice of quality functional optimized, index (4) is relevant. In this paper, the connection of dynamical loading during a turn with the coefficients of quality index was found. The minimizing the optimized functional leads to minimum of the integral loads, and the reorientation is executed maximum fast and so that the construction loading (due to spacecraft rotation) would not exceed a certain preset value (dynamical loads is specified by the coefficient k_0 only). The example of numerical decision of rotation problem and mathematical modeling that demonstrates the behavior of spacecraft motion parameters are given. The efficiency of the suggested control of spacecraft reorientation is determined by a considerable reduction of maneuver duration under bounded dynamic loads due to rotation of the spacecraft around its center of mass. This is important in the cases when the kinetic factor turns out to be crucial for normal functioning of a spacecraft and its onboard systems, for turns through large angles and when the Euler axis does not lie in the spacecraft's transverse plane and does not coincide with its longitudinal axis.

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