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Symmetries of *p*-Polygons

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Abstract
In addition to general considerations, the present work includes the enumeration of the equivalence-classes of p-polygons with p vertices for p ≥ 5 with certain symmetry properties:
1. We count the equivalence-classes of p-polygons with p symmetry axes, the so-called regular polygons.
2. We count the equivalence-classes of p-polygons with exactly one axis of symmetry.
3. We count the equivalence-classes of p-polygons with no axis of symmetry, the so-called asymmetrical p-polygons.
For p = 5 and p = 7 we show in all three cases a set of representatifs of the equivalence classes.

Keywords: Hamiltonian cycles, p-Polygons, Symmetry

1. Introduction
Polygons are among the most fundamental objects of Euclidean geometry and have been studied since antiquity. Beyond their geometric significance, they have also appeared in cultural and symbolic contexts, for example, the pentagram was regarded by the Pythagoreans as a figure of harmony and healing. From a mathematical perspective, classical investigations have focused on properties such as angles, diagonals and area. Euler, using the totient function, derived and proved a formula for the number of distinct regular polygons that can be formed by placing n equidistant vertices on a circle. In the twentieth century, Golomb and Welch extended the problem to polygons in general, where the vertices need not to form regular shapes [1]. They considered polygons equivalent if they differed only by a rotation in the plane, and similar if they could also be related by reflection through an axis. Using a combinatorial and group-theoretic approach, based on the action of the dihedral group D_n and Burnside's Lemma, they obtained formulas for the number of distinct equivalence-classes of n-polygons. Their work thus provided a complete enumeration of fundamentally distinct n-polygons. However, their treatment did not address the degree of symmetry of a polygon: all symmetric and asymmetric cases were treated uniformly, without finer classification. The study of symmetry degrees in polygons is of intrinsic interest for several reasons: Symmetry is a central organizing principle in mathematics, with implications ranging from group theory to crystallography. In polygons, the degree of symmetry can be defined in terms of the size of the subgroup of the dihedral group D_n, that leaves the polygon invariant. At one extreme, regular polygons exhibit maximal symmetry, while at the other extreme, generic polygons are completely asymmetric, invariant only under the identity transformation. Between these extremes lie intermediate cases of partial symmetry. A systematic classification of polygons according to these symmetry degrees enriches our understanding of the combinatorial structure of polygons and provides insight into the balance between order and disorder in geometric configurations. In earlier works I investigated two highly symmetric cases and discovered an unexpected connection to perfect numbers [2,3]. These case studies suggested deeper structural relationships but did not yield a general framework for classifying polygons by symmetry. The aim of the present paper is to develop such a framework for polygons with p vertices, where p is a prime number greater than 3. The restriction to prime numbers is natural, since the symmetry structure is considerably simpler in the prime case, while composite values of n introduce additional complications. This allows for a clear and complete classification, which can then serve as a foundation for the more general case of arbitrary n.

1.1. Definition of a *p*-polygon
Let be $p \geq 5$ a prime number. p vertices are regularly distributed in a circle. We consider the Hamiltonian cycles through the p vertices. In this paper such Hamiltonian cycles are called p-polygons. The usual polygons are the special case where all edges have minimal length.

Let p be a natural number $p \geq 5$ and $S^1 \subset \mathbb{R}^2 = \mathbb{C}$ the unit circle in the Euclidean plane. The finite subset

$$V_p := \{v_k := e^{2\pi ik/p} \mid k = 0, 1, \ldots, p-1\} \subset S^1$$

represents the vertices of a p-polygon.
To describe the p- polygons we use the p-cycles $\sigma = (\sigma_1,\sigma_2,\cdots,\sigma_p)$ consisting of the p numbers $\{0,1,\cdots,p-1\}$ in any order. The associated p-polygon $P(\sigma)$ is given by the path $v_{\sigma 1}v_{\sigma 2}\cdots v_{\sigma p}v_{\sigma 1}$ or more precisely by combining the links $v_{\sigma i}v_{\sigma i}+1$, $i = 1,2,\cdots,p$, where $\sigma_p+1 = \sigma_1$.

1.2. Definition of the used equivalence relation

We denote by $C(p)$ the set of all p-polygons and define the following equivalence relation on $C(p)$:
Two p-polygons $P_1(p)$ and $P_2(p)$ are said to be equivalent, denoted $P_1(p) \equiv P_2(p)$, if they are obtainable from one another by a rotation. This means, that the two considered p-polygons have the same shape and belong to the same equivalence-class.

Explanation: Two p-polygons $P_1(p)$ and $P_2(p)$ are not equivalent, denoted $P_1(p)/\equiv P_2(p)$ if they are not obtainable by a rotation. But there exists a reflection and their shapes are mirrored, but not congruent. Therefore, the two considered p-polygons have not the same shape and therefore they belong to different equivalence-classes.

1.3. Example for Similar but Not Equivalent Polygons

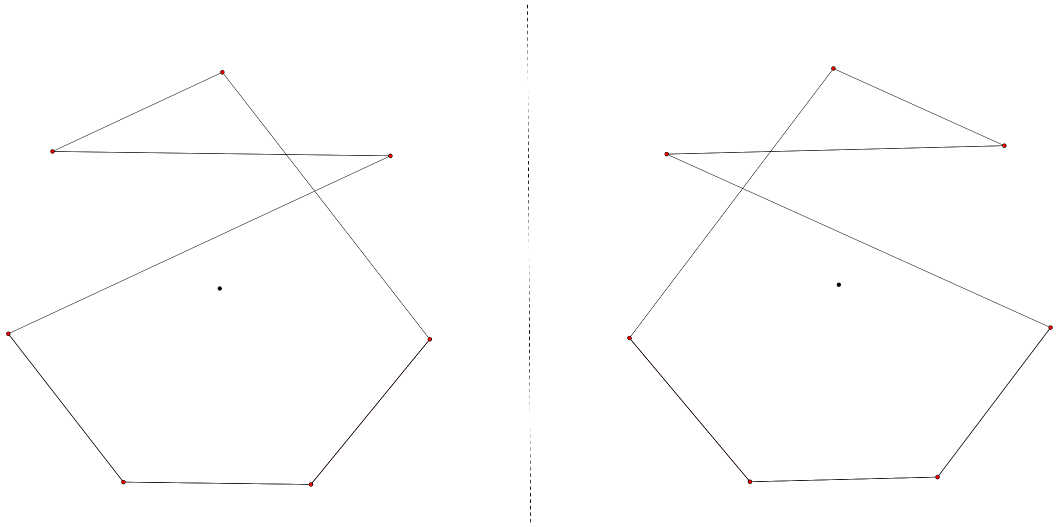

Figure 1: Representatifs of Two Equivalence Classes

1.4. The Question to Deal with in This Article

As $p \geq 5$ is a prime number and every prime number has the only divisors are 1 and p, there exist only three different kinds of equvalence-classes of p-polygons, i.e. three different kinds of shapes:

- Equivalence-classes with p symmetry-axes (socalled regular or star-polygons)
- Equivalence-classes with 1 symmetry-axis
- Equivalence-classes with no symmetry-axis, i.e. equivalence-classes of asymmetrical p polygons.

We prove the numbers:

- $|X_p(p)|$ of the equivalence-classes with p symmetry-axes
- $|X_1(p)|$ of the equivalence-classes with 1 symmetry-axis
- $|X_0(p)|$ of the equivalence-classes with no symmetry-axis

In addition, we illustrate the results by a set of representatifs of these equivalence-classes in all three cases for the prime number $p = 5$ and $p = 7$

2. Results

2.1. Number of Equivalence-Classes of all P-Polygons

Let be $p \geq 5$ a prim number.

$$|P(p)| = \frac{(p-1)! + (p-1)^2}{2p}$$

2.2. The Regular P-Polygons

Let be $p \geq 5$ a prim number. The number $|X_p(p)| = \frac{\varphi}{2} = \frac{p-1}{2}$. $\varphi(p)$ denotes the Euler divisor function of p.[Cox73]

2.3. P-Polygons With at Least 1 Symmetry-Axis and with Exact 1 Axis

Let be $p \geq 5$ a prim number. A p-polygon with at least 1 symmetry-axis has either 1 or p symmetry-axes. $|X_{1+}(p)|$ is the number of the equvalence-classes of such p-polygons. $|X_1(p)|$ is the number of the equvalence-classes of p-polygons with exact 1 axe of symmetry.

$$|X_{1+}(p)| = \frac{p-1}{2} \cdot 2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}\right)!$$

$$\boxed{|X_1(p)| = \frac{p-1}{2} \cdot [2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}\right)! - 1]}$$

2.4. The Asymmetrical P-Polygons

$$|X_0(p)| = \frac{(p-1)^2 + (p-1)! - p \cdot (p-1) \cdot 2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}\right)!}{2p}$$

3. Proofs

3.1 Number of Equivalence-Classes of all P-Polygons

This number $|X(n)|$ is proved by S.W.Golomb and L.R.Welch for odd values of n:

$$|X(n)| = \frac{1}{2n^2} \left(\sum_{d|n} \varphi^2 \left(\frac{n}{d}\right) \cdot d! \cdot \left(\frac{n}{d}\right)^d \right),$$

where d are divisors of n and φ denotes the Euler divisor function.

Let be $p \geq 5$ a prime number, then the divisors of p are only 1 and p and the formula of Golomb and Welch changes into:

$$|X(p)| = \frac{(p-1)! + (p-1)^2}{2p}$$

3.2. Number of Equivalence-Classes of the p-Polygons With at Least One Axe

Proof. A representatif for each equivalence-class of p-polygons with at least one axe of symmetry will get constructed using the following construction figure:

$v_1, v_2, \ldots v_i, \ldots v_{\frac{p-2}{2}}, v_{\frac{p-1}{2}},$

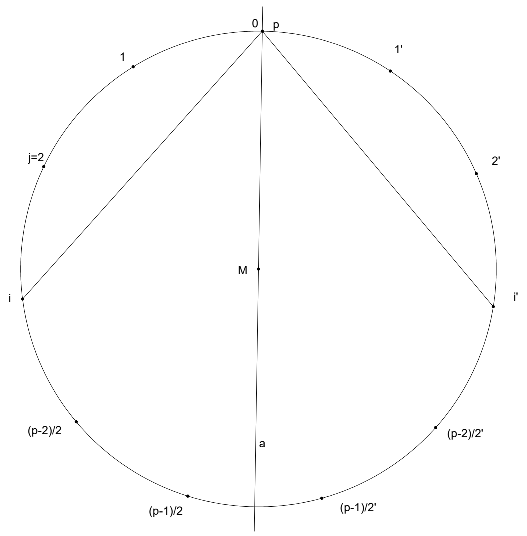

Figure 2: Construction-Figure 1

The representatifs start and end in the vertex labeled with 0 and p, i.e. $v_0 = v_p$. One axe is vertical and labeled with a. The $\frac{p-1}{2}$ left-handed vertices are $v_1, v_2, \ldots v_i, \ldots v_{\frac{p-2}{2}}, v_{\frac{p-1}{2}}$, while the right-handed vertices are $v_{1'} = v_{p-1}, v_{2'} = v_{p-2}, \ldots v_{i'} = v_{p-i} \ldots v_{\left(\frac{p-2}{2}\right)'} = v_{\frac{p+2}{2}}, v_{\left(\frac{p-1}{2}\right)'} = v_{\frac{p+1}{2}}$. We choose a first pair of vertices, which are situated sym metrically to the axe a , e.g v_i and $v_{i'}$. We connect those two vertices with the vertex $v_0 = v_p$. So we get an open chain $C(3)$ of three vertices: $C(3) = v_i v_0 v_{i'}$.

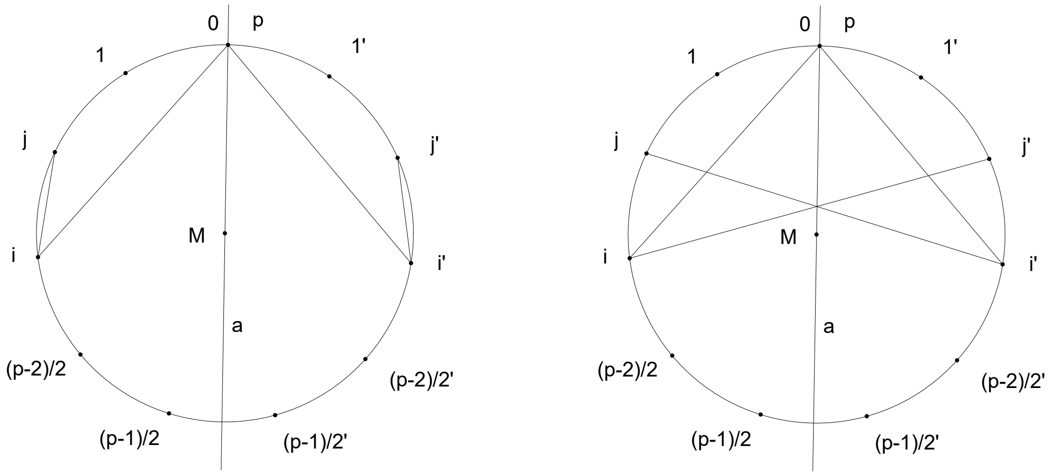

Figure 3: Construction Figure 2

If there exist vertices not included in $C(3)$ we choose a pair of vertices v_j and $v_{j'}$ with $j \neq i$. We have two possibilities to connect v_j and $v_{j'}$ with the ends v_i and $v_{i'}$:We get two open chains of 5 vertices:$C_1(5)$ = $\overline{v_j v_i v_0 v_{i'} v_{j'}}$ and C2(5)) = $\overline{v_{j'} v_i v_0 v_{i'} v_j}$. If $p = 5$ there exist no vertices which are not yet included into the construction and we finish the construction by connecting $v_{j'}$ and v_j to get closed chains of 5 vertices, representing the equivalence-classes of the 5-polygons with at least one symmetry axe.

If $p > 5$ we repeat the construction until all p vertices are included and finally close the chains to get the p-polygons with at least one axe of symmetry by connecting the two vertices, which were involved into the chains $C(p)$ in the last step.

Counting the Possibilities:

1. Choosing the first pair of vertices: $\frac{p-1}{2}$ possibilities
2. Choosing the second pair of vertices: $\frac{p-3}{2}$ possibilities 2
3. Connecting the second pair with the first pair: 2 possibilities
4. The number of chains of 5 vertices: $|C(5)| = \frac{p-1}{2} \cdot \frac{p-3}{2} \cdot 2$
5. Choosing the third pair of vertices: $\frac{p-5}{2}$ possibilities
6. Connecting the third pair with the second pair: 2 possibilities
7. ...
8. Choosing the last pair of vertices: $1 = p - (p-1)$ possibility
9. Connecting the last pair of vertices with the second to last: 2 possibilities
10. Closing the chain: 1 possibility

In general:$|P_{1+}(p)| = 2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}!\right) \cdot \left(\frac{p-1}{2}\right)$
Due to Leonhard Euler:$|P_p(p)| = \frac{\varphi}{2} = \frac{p-1}{2}$. we conclude:

$$\boxed{|P_1(p)| = \frac{p-1}{2} \cdot \left[2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}\right)! - 1\right]}$$

3.3. Number of the Equivalence-Classes of Asymmetrical p-Polygons

$$|P_0(p)| = |P(p)| - |P_{1+}(p)| \text{ Also:} |P_0(p)| = \frac{(p-1)! + (p-1)^2}{2p} - 2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}\right)! \cdot \left(\frac{p-1}{2}\right)$$

It follows:

$$\boxed{|X_0(p)| = \frac{(p-1)^2 + (p-1)! - p \cdot (p-1) \cdot 2^{\frac{p-3}{2}} \cdot \left(\frac{p-3}{2}\right)!}{2p}}$$

4. Table of the Results

p	$\lvert P(p)\rvert$	$\lvert P_r(p)\rvert$	$\lvert P_1(p)\rvert$	$\lvert P_0(p)\rvert$
5	4	2	2	0
7	54	3	21	30
11	164950	5	1915	163030
13	18423144	6	23034	1840010

Table 1: Result for $5 \leq p \leq 13$

5. Representatifs of the Equivalence-Classes

5.1. Representation of the 5-Polygons

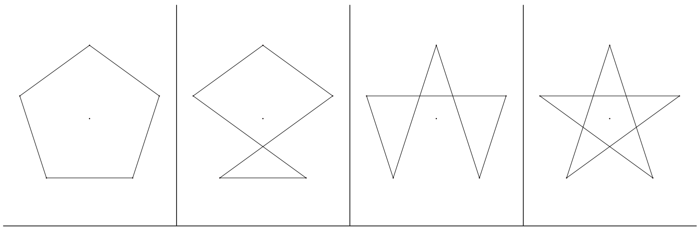

Figure 4: Representatifs of the 4 Equivalence-Classes of the 5-Polygons

5.2. Representation of the Asymmetrical 7-Polygons

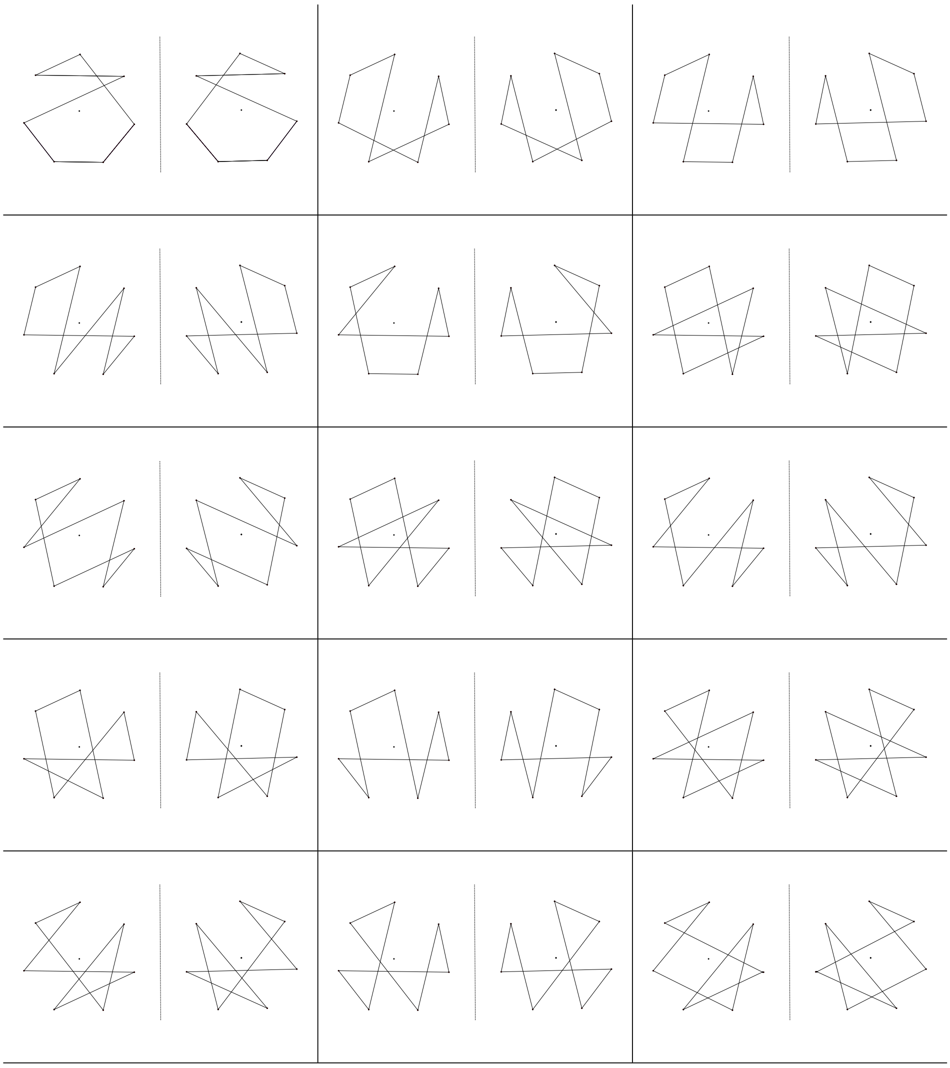

Figure 5: Representatifs of the 30 Equivalence-Classes of Asymmetrical 7-Polygons

Between the two mirrored representatifs is always drawn a small vertical line. They belong to two different equivalence-classes.

5.3. Representation of the 24 Symmetrical 7-Polygons

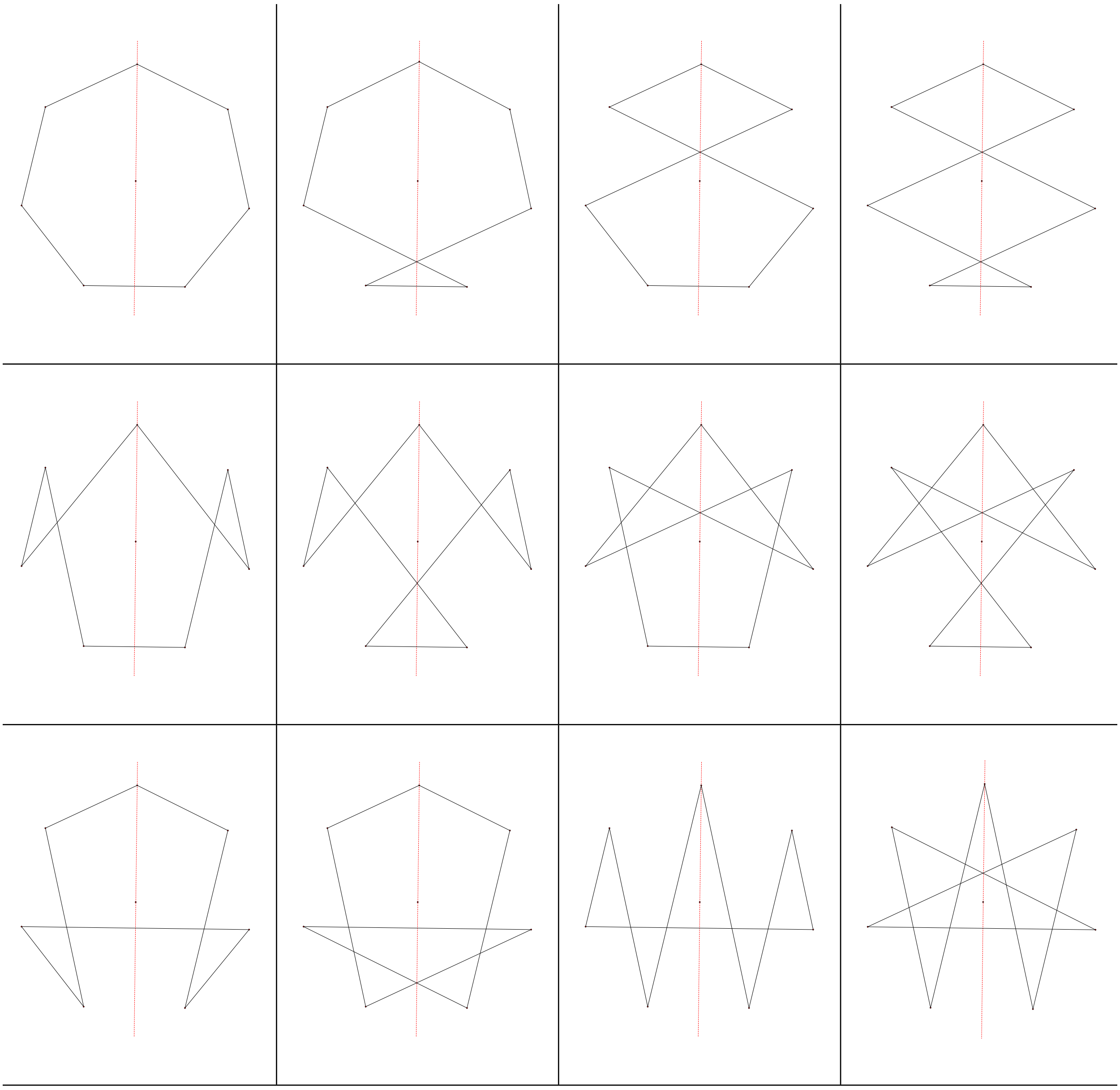

Figure 6: Representatifs of 12 Equivalence-Classes of the 7-Polygons With At Least One Axe

These symmetrical figures have a vertical axe and represent 12 equivalence-classes.

5.4. Representation of the 24 Symmetrical 7-Polygons (Fortsetzung)

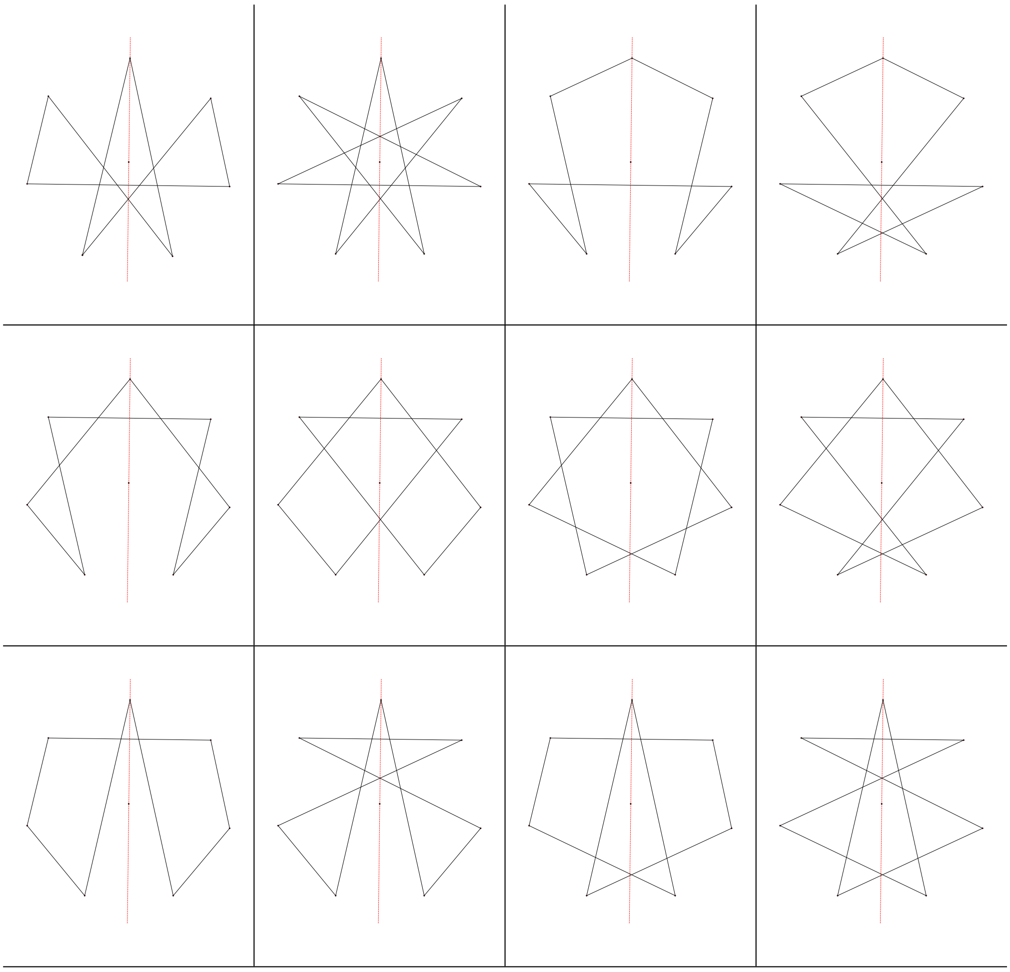

Figure 7: Representatifs Of the Other 12 Equivalence-Classes of the 7-Polygons With At Least One Axe

These symmetrical figures have also a vertical axe and represent the other 12 equivalence-classes.

List of Figures

List of Tables

Acknowledgments

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