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Review Article

The Exact Formula for the Ellipse Perimeter by Using Geometric Constructive Methodology

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Abstract

For many years since the ellipse was introduced to the mathematical world. Many theorems has been developed including formulas, but we could not come up the equation for the ellipse perimeter. We use the integral and numerical analysis methods to approximately measure the ellipse perimeter. The best known approximation formula of Sir. Srinivasa Ramanujan is consider to be the best and has most accurate result. This paper will demonstrate how I use geometry constructive method to establish the exact equation for the ellipse perimeter.

Keywords: Major axis, Minor axis, Semi-major axis, Semi-minor axis, Projected angle, Transformation angle, Equator tangent line, Isosceles Triangle, Pythagoras Theorem, Parallelogram.

1. Introduction

Since Kepler discover the ellipse orbit in the stars. Through the study of the ellipse geometry, we came up with many formulas for the area of the ellipse and others, except the formula for the ellipse's perimeter. There are several approximation formulas, but most of them are using reasoning outcome conditions to establish some kind of formula then check its results by verifying with the results from the approximation formula of Sir. Srinivasa Ramanujan. which is considered to be the best and most accurate results. But if there is no real exact result how could it determined the errors in the result. While working on my paper about the relation between the circle and the ellipses. I came up a way how to calculate the perimeter of an ellipse which is based on basic geometric constructive method to come about the geometric equation.

1.1. The Ellipse Perimeter Equation Theorem

The ellipse with Semi major axis is a, and Semi minor axis is b. Then its perimeter P is calculated by the following formula.

$$P = 4a\sqrt{1 + (b/a)^2((\pi/2)^2 - 1)}$$

To prove this, it would need to establish the following theorems

Theorem 1

If a plane cut crossing the cylinder at an angle then the intersected curve line of the plan cutting through the cylinder is an ellipse.

Proof

Let's have the cylinder tangent to the equator of an inscribed sphere. And M is the equator plane cut through the equator tangent line of the sphere at bisector of the cylinder as in figure 1a.

Let N is an inclined plane cuts through the cylinder and the sphere passing through the centroid O of the sphere and the cylinder at an angle theta as in figure 1b

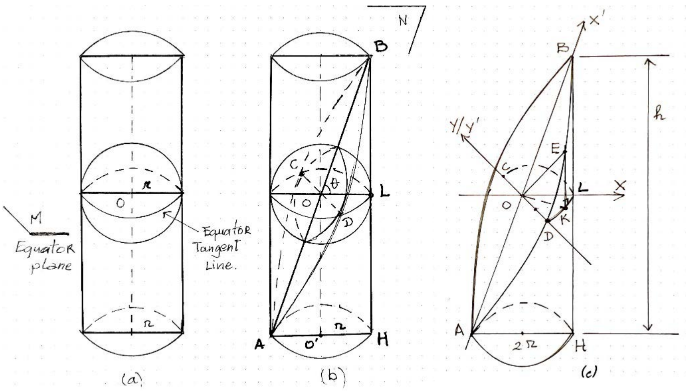

Figure: 1

Due to the Symmetric and Parallelism Condition, the Following Facts are established.

OL = AO'
BL = LH = OO' = h/2
AO = OB = a
CO = OD = OL = r
The two right triangle angle △ BOL and △OAO' are congruence Cos (Theta) = cos (*L* BOL) = r/a (1)

Figure 1c illustrates the Intersected Closed Loop Curve Line Of The Plane N And The Cylinder. Let E is a Point on This

Intersected Closed Loop Curve Line. And K Is the Projected Point of E onto the Equator Tangent Line of the Sphere (figure 1c).

E(x',y') Cartesian coordinate of E on the plane N
K(x,y) Cartesian coordinate of K on the plane M
Then
$y = y'$
$x = x'\cos(\text{Theta})$
$OK = OL = r$
$EK = x'\sin(\text{Theta})$

Polar parameter equation for E would be on the plane N would be

$$x'^2 + y'^2 = \overline{OE}^2$$

$$\overline{OE}^2 = \overline{OK}^2 + \overline{EK}^2 \qquad \text{Pythagoras on } \triangle EKO$$

$x'^2 + y'^2 = r^2 + x'^2\sin^2(\text{Theta})$
$= r^2 + x'^2(1-\cos^2(\text{Theta}))$
$= r^2 + x'^2 - x'^2\cos^2(\text{Theta})$
$x'^2\cos^2(\text{Theta}) + y'^2 = r^2$
$x'^2(r/a)^2 + y'^2 = r^2$ Substitute $\cos^2$ (Theta) with equation (1) above
$x'^2/a^2 + y'^2/r^2 = 1$

It's the polar parameter equation of the ellipse. So it proves that intersected curve line of the plan cut through the cylinder is an ellipse with
Semi-major axis = a
Semi-minor axis = b = r (The radius of the cylinder)

Corollary 1

If wrapping an isosceles triangle into a cylinder which its circumference is equal to the base length of the triangle then the two sides would form an ellipse.

Proof

Let's have two congruence isosceles triangles △ AB"A' and △ BA"B' and C, D, C', D' are mid Points of the Sides as Shown in Figure 2.

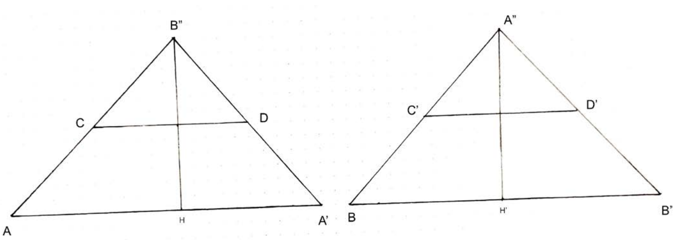

Figure: 2

Let's Rotate 180° one of Them and Attach one Side Together as in Figure 3. It Forms a Parallelogram ab'ba because both Pairs of Opposite Sides are Equal.

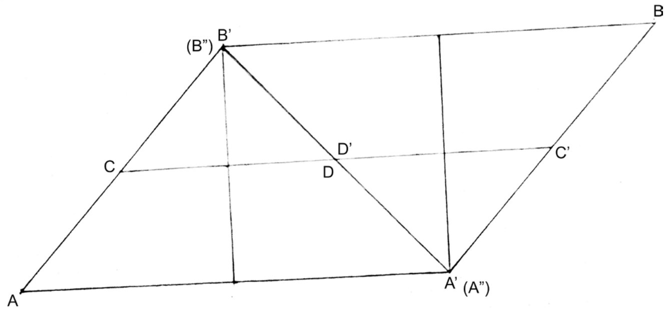

Figure: 3

Let's Wrap This Parallelogram Around and Attach the Other Two Sides, It Then Forms A Cylinder as In Figure 4a. It's Similar to the One in Figure 1b Above. Since The Height of any Point on the Side Does Not Changes and All Project Down to Base Which Now is a Circle as a Condition in Theorem 1 Above. So the two Sides do Form an Ellipse as the Theorem 1 Above.

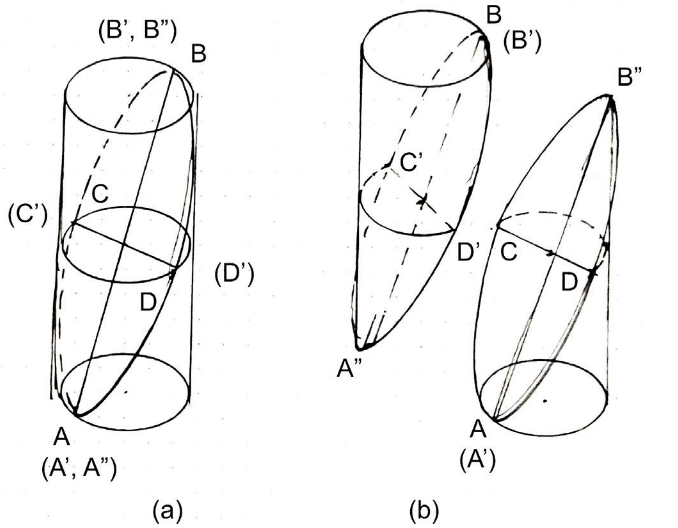

Figure: 4

If we detach the cylinder as two part we will have two objects as in figure 4b which is similar to figure 1c. Back to the figure 1b, after cutting the cylinder with the plane N then detach them into two identical objects as in figure 5a below. Again, it's similar to figure 4b above.

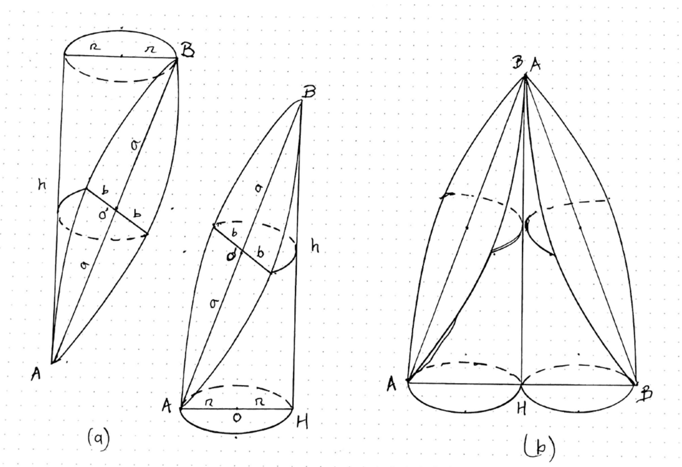

Figure: 5

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If you put these two parts against each other as in figure 5b and rotate them against the side then they will match to each other as a mirroring image because their dimensions are the same.

If we disconnect the 3D shape at A and B at the base of these two objects, and spreading flat down to transform them onto a 2D plane, then we will have a isosceles triangles ABA' and BAB' as in figure 6. The reason for the AB and BA' are straight line because. If the sector AB is curved then if this sector is convex on one object, then it has to be concave on the other subject or vice versa then they won't be identical. So AB and BA' have to be straight line. It does concur with the corollary 1 above.

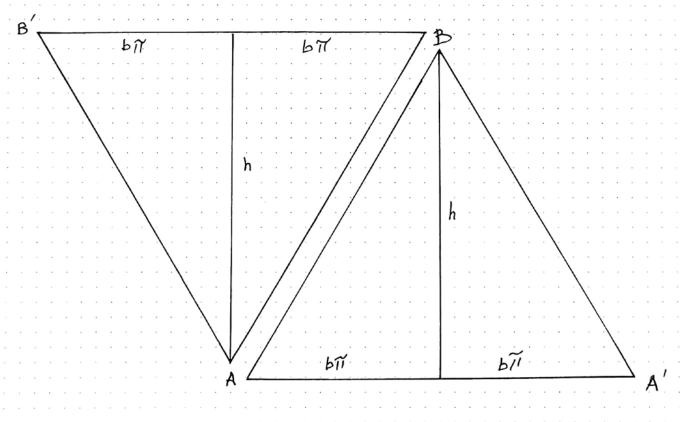

Figure: 6

1.2. The perimeter of the ellipse formula

Simplifying this transformation could be presented as figure 7a to 7b below

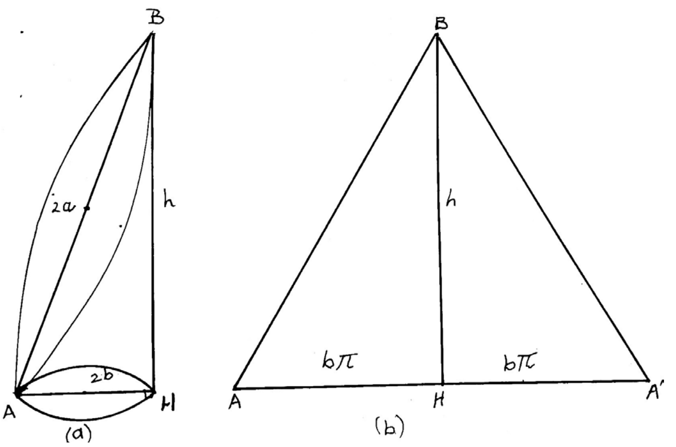

Figure: 7

Then the perimeter of the ellipse equivalent to the sum of side length AB and BA'. So the perimeter of the ellipse P could be expressed as.

P = AB + BA' = 2AB

BH = h Regard to figure 7a or 7b, both are the same.

So the perimeter of the ellipse would be:

$P = 2\sqrt{b^2Pi^2 + h^2}$ (2) Pythagoras theorem based on △ AHB in Figure 7b

If / Theta = 0 then h = 0

So P = 2b x Pi

It concurs with the special ellipse condition which the semi major axis and the semi minor axis are concurrence. It's the circumference of the circle with the radius b of the cylinder.

Since

$h^2 = 4a^2 - 4b^2$ Pythagoras theorem based on △ AHB in figure 7a

Substitute h^2 into perimeter equation (2) above we have

$P = 2\sqrt{b^2Pi^2 + 4a^2 - 4b^2}$

$P = 2\sqrt{4a^2 + b^2(Pi^2 - 4)}$

$P = 2\sqrt{4a^2(1+ (b^2/a^2)(Pi^2 - 4)/4}$

$P = 4a\sqrt{1+ (b/a)^2((Pi/2)^2 - 1)}$

This could be used as an exact equation to calculate the ellipse perimeter. Again it concurs a special case when both semi-major and minor axis are concurrence then P=2a x Pi It's the circle's circumference.

As if the ratio of semi minor axis and semi major axis is approaching zero as long as it is greater than zero then the equation does show the perimeter of the ellipse also approaching 4a.

Let "b" could not equal zero because that would be a line not an ellipse. But for the shake of argument if the circumference is considered as a complete loop, where it is the distance where you start at one point and return back to the same point, then it would be considered correct. P is 4a even it is a straight line.

Just for the reference, following is the table and plot of my equation and Sir. Srinivasa Ramanujan approximation. It's based on "a" equal 1 and the ratio b/a (note that this is the cosine of the angle theta, which I have written about this angle in my study of the ellipse).

For Pi = 3.141592653			
	b/a	HS Geometric Equation	Ramanujan Approximation
1	0	4	4
2	0.01	4.000293469	4.00109833
3	0.1	4.029241141	4.06397418
4	0.2	4.115718249	4.20200891
5	0.3	4.255943795	4.38591007
6	0.4	4.444833722	4.60262252
7	0.5	4.676494884	4.84422411
8	0.6	4.944919649	5.10539977
9	0.7	5.244466095	5.38236898
10	0.8	5.570115552	5.67233358
11	0.9	5.917560161	5.97316043
12	1	6.283185306	6.28318531

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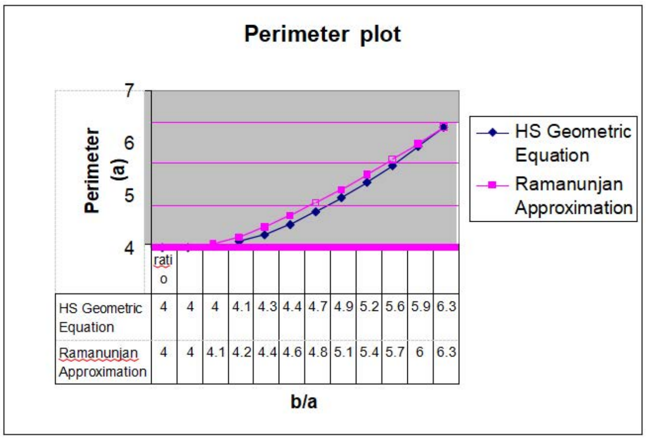

2. Conclusion

With the above basic geometric constructive method, I proved that the exact formula for the ellipse perimeter is.

$$P = 4a\sqrt{1 + (b/a)^2((\pi/2)^2 - 1)}$$

Notes: All of the figures drawing do not reflect the accuracy of the image dimension and shape. They are all estimated to illustrate the calculation concept.

References

1. Green, S., L. (2006). Lecture Notes on Coordinate Geometry: Ellipse Reference: Advance Level Pure Mathematics p.72-87 2-5. Wolfram Mathematic-Ellipse (Web site).
2. Clynch, J. R., & Garfield, N. (2006). Equations of an Ellipse. San Francisco State University, San Francisco, 14.
3. Ellipse's Perimeter Estimations By Necat Tasdelen, necattasdelen@ttmail.com.HANDBOOK OF MATHEMATICAL TABLES ANDFORMULAS By Richard Stevens Burington, Ph.D. (McGRAW-HILL BOOK COMPANY).