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The Power Problem for Generalized Gamma Convolutions (GGC) and Related Questions

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Abstract

The class of generalized gamma convolutions (GGC) is closed with respect to change of scale, weak limits and addition and multiplication of independent random variables. Our main result confirms an old conjecture that GGC is also closed wrt q−th powers, q > 1. The proof uses explicit iterative formulas for the densities of finite sums of independent gamma variables, hyperbolically completely monotone functions (HCM) and the Laplace transform. We apply the result to sums and products of q−th powers of independent GGCs, q ≥ 1, symmetric extended GGC (symEGGC) and a new proof that X ~ GGC implies Exp(X) ~ GGC.

Keywords: Gamma Distribution, Generalized Gamma Convolution (Ggc), Completely Monotone (Cm), Hyperbolically Completely Monotone (Hcm), Bernstein Function, Laplace Transform

1. Introduction

The generalized gamma convolutions (GGC) were introduced by O. Thorin in his study of infinite divisibility of the lognormal distribution, see also [10,11,12]. The class GGC consists of limit distributions of finite sums of independent gamma random variables (rvs) and is closed with respect to (wrt) change of scale, weak limits and sums and products of independent rvs. A comprehensive study of GGC and its relation to hyperbolically completely monotone functions (HCM) is found in Bondesson, see also Steutel, van Hahn, Ch. VI, §5 and Bondesson [3,4,9]. We use Feller as a general reference on probability theory [6]. For more on the background in infinite divisibility, GGC and the pioneering work of O. Thorin, see the nice biography by Bondesson, Grandell, Peetre [5]. A problem on a class of mixtures of gamma distributions in the same field was studied in Behme, Bondesson and by the author in [2,8].

Our main result (Theorem 1) confirms an old conjecture for GGC going back at least to the late 1980's, that $X \sim GGC$ and $q > 1$ implies that $X^q \sim GGC$ (here called the Power Problem) mentioned in Bondesson, p. 97 [3]. It is known to hold in several special cases, see Bondesson, Ch.6 and, Sec.7 [3,4]. If PF_∞ denotes the class of limit distributions of finite sums of exponential rvs, then $X \sim PF_\infty$ implies that $X^q \sim GGC$ for $q \geq 1$, Bondesson, Theorem 6.2.7 [3]. The conjecture is then also true for sums of independent gamma rvs whose shape parameters are positive integers. A positive answer to the Power problem was conjectured in Bondesson, Conjecture 1 [4]. Our main result (Theorem 1) confirms the conjecture. The proof is based and Bondessons characterization of GGC in, Theorem 5.4.1, explicit iterative formulas for the densities of finite sums of independent gamma rvs and successive substitutions [3]. The result is applied to a new proof that $X \sim GGC$ implies that $e^X - 1 \sim GGC$, Bondesson, Theorem 4, (Theorem 2), to sums and products of powers of independent GGCs (Theorem 3) and inclusion theorems for symmetric extended GGCs, $symEGGC$ (Theorem 4) [4]. Section 2 begins with the standard notation used in this field, a review of our set up and three lemmas, where Lemma 2 is used in the induction step of the proof. Our main result (Theorem 1) is stated and proved in Section 3 and the applications are given in Section 4.

2. Background

This section gives the necessary background and defines the concepts needed to state and prove our theorems, c.f. Bondesson [3]. A function $f : (0,\infty)^n \to [0,\infty)$ is completely monotone (CM) if $(-1)^m D^m f \geq 0$, for all positive integers m, and a function $f : (0,\infty) \to [0,\infty)$ is hyperbolically completely monotone (HCM) if, for every fixed $u > 0$, $H(w) = f(uv) \bullet f(u/v)$ is CM wrt $w = v + v^{-1}$, see Bondesson [3], Ch. 5. We let $Gamma(\beta,b)$ denote a standard gamma distribution with density $f(x) = b^\beta \bullet \Gamma(\beta)^{-1} \bullet x^{\beta-1} \bullet e^{-bx}$, $x > 0$, and write $Gamma(1,b) = Exp(b)$ for the exponential distribution. A generalized gamma convolution (GGC) is defined as a limit distribution of finite sums $X_1 + X_2 + \bullet\bullet\bullet + X_n$ of independent gamma rvs $X_i \sim Gamma(\beta_i, b_i)$, $1 \leq i \leq n$. Then $X \sim GGC$ if and only if the Laplace transform φ of the distribution of X can be represented as

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$$\phi(s) = E[e^{-sX}] = \exp\big(-as + \int \log(\frac{t}{t+s})\,U(dt)\big),\ s \geq 0,$$

where $a \geq 0$ (called the left extremity) and $U\,(dt)$ is a nonnegative measure on (0,∞), with finite mass on compact subsets of (0,∞), such that $\int_0^1 |\log t|\,U(dt) < \infty$ and $\int_1^\infty t^{-1}\,U(dt) < \infty$, Bondesson [3] Ch. 3. We use the following wellknown characterization of GGC, see Bondesson [3], Theorem 5.4.1.

Proposition 1 *A function* $\varphi(s)$ *defined on* (0,∞) *such that* $\varphi(0+) = 1$ *is the Laplace transform of a GGC if and only if* φ *is HCM.*

We begin our analysis of q−th powers of *GGC* and the proof that $X \sim GGC$ implies $X^q \sim GGC$ by considering finite sums $S_n = X_1 + X_2 + \bullet\bullet\bullet + X_n$ of independent gamma rvs, $X_i \sim Gamma\ (\beta_i, b_i)$, $1 \leq i \leq n$. We recall that if all $\beta_i = 1$, $1 \leq i \leq n$, then S_n is a sum of independent exponentially distributed rvs and $X^q \sim GGC$ by Bondesson, p. 96 [3]. The same conclusion follows if all β_i are positive integers, since then each X_i is a sum of β_i independent Exp(b_i) rvs and we are back in the first case. Since *GGC* is closed wrt weak limits it is no loss of generality to assume that each β_i is a rational number $\beta_i = p_i / N$, for some positive integer p_i and a common denominator $N \geq 2$, $1 \leq i \leq n$. Then each X_i is the sum of p_i independent rvs $X_{i,j} \sim Gamma(1/N, b_i)$, $1 \leq i \leq n$, $1 \leq j \leq p_i$. The Laplace transform φ_{S_n} of Sn becomes

$$\phi_{S_n}(s) = \prod_{i=1}^{n} (\frac{b_i}{s+b_i})^{p_i/N} = \prod_{i=1}^{n} \Big(\prod_{j=1}^{p_i} (\frac{b_i}{s+b_i})^{1/N} \Big)$$

and S_n is a finite sum of independent gamma distributed rvs with form parameter $1/N$, by the uniqueness of the Laplace transform, Lemma 3. It is thus no loss of generality to assume that $\beta_i = \beta$, $1 \leq i \leq n$, for some $\beta > 0$.

We start with the formulas for the density of the sums S_n above in their most general form and specialize to the case $\beta_i = \beta$, $1 \leq i \leq n$, later. When $n = 2$, a direct calculation gives

$$f_{S_2}(x) = \frac{b_1^{\beta_1} \cdot b_2^{\beta_2}}{\Gamma(\beta_1) \cdot \Gamma(\beta_2)} \cdot x^{\beta_1+\beta_2-1} \cdot \int_0^1 e^{-x\cdot(b_1(1-u)+b_2 u)} \cdot (1-u)^{\beta_1-1} \cdot u^{\beta_2-1}\, du \quad (1)$$

and for a general n we use Akkouchi [1], Theorem 1 to get

$$f_{S_n}(x) = D_n \cdot x^{\beta_1+\beta_2+\cdots+\beta_n-1} \cdot \int_0^1 \cdots \int_0^1 e^{-x\cdot C_n(\mathbf{u})} \cdot B_n(\mathbf{u})\, du_1 du_2 \cdots du_{n-1}, \quad (2)$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_{n-1})$ and

$$C_n(\mathbf{u}) = b_1(1-u_1) + b_2 u_1(1-u_2) + \cdots + b_{n-2} \cdot u_1 u_2 \cdots u_{n-3} \cdot (1-u_{n-2}) +$$

$$+ b_{n-1} \cdot u_1 u_2 \cdots u_{n-2} \cdot (1-u_{n-1}) + b_n \cdot u_1 u_2 \cdots u_{n-1},$$

We note that the sum of last two terms in $C_n(\mathbf{u})$ simplies to

$$u_1 u_2 \cdots u_{n-2} \cdot \big(b_{n-1} \cdot (1-u_{n-1}) + b_n \cdot u_{n-1}\big), \quad (3)$$

which is used in the proof of Theorem 1 for $n \geq 3$. Further

$$B_n(\mathbf{u}) = \frac{\Gamma(\beta_1+\beta_2+\cdots+\beta_n)}{\Gamma(\beta_1)\Gamma(\beta_2)\cdots\Gamma(\beta_n)} \cdot \prod_{j=1}^{n-1} u_j^{\beta_1+\beta_2+\cdots+\beta_j-1} \cdot (1-u_j)^{\beta_{j+1}-1}$$

and

$$D_n = \frac{b_1^{\beta_1} \cdot b_2^{\beta_2} \cdots b_n^{\beta_n}}{\Gamma(\beta_1+\beta_2+\cdots+\beta_n)},$$

for all $\mathbf{u}$.

Lemma 1 (Feller [6], Criterium 2, p. 441) *Let* $f : (0,\infty) \to (0,\infty)$ *be CM and assume that* $g : (0,\infty) \to (0,\infty)$ *has a CM derivative. Then* $f \circ g$ *is CM.*

The next lemma is used in the induction step of the proof of Theorem 1.

Lemma 2 *Let* b_1, b_2, β_1, β_2, *A and B be positive numbers and* $0 < \alpha < 1$. *Then*

$$L = \int_0^1 \int_0^1 e^{-E} \cdot g(u,v)\, du\, dv, \tag{4}$$

where $g = g(u,v) = \big((1-u)(1-v)\big)^{\beta_1 - 1} \cdot (uv)^{\beta_2 - 1}$ *and*

$$E = b_1\big((1-u)Ay^{\alpha} + (1-v)By^{-\alpha}\big) + b_2(uAy^{\alpha} + By^{-\alpha})$$

is CM wrt $Ay^{\alpha} + By^{-\alpha}$, *for* $y > 0$.

The proof of Lemma 2 for $n = 2$ is contained in the proof of Theorem 1. The general case is proved at the end of the next section.

Lemma 3 (Feller [6], Chap. XIII.I,Theorem 1, p. 408) (*Uniqueness.*) *Distinct probability distributions have distinct Laplace transforms.*

3. Powers

In this section we state and prove our main result that GGC is closed wrt taking q−th powers, $q > 1$.

Theorem 1 *Let* $q > 1$ *and assume that* $X \sim GGC$. *Then* $X^q \sim GGC$.

Proof. As noted above, it is enough to prove the theorem for finite sums $S_n = X_1 + X_2 + \bullet\bullet\bullet + X_n$ of independent gamma variables, where $X_i \sim Gamma(\beta, b_i)$, $1 \le i \le n$, for all $\beta > 0$. The proof is by induction over n and uses explicit formulas for the density fs_n of S_n and the Laplace transform $\phi_{S_n^q}$ of S_n^q. We start with the case $n = 2$. Then the density of S_2 is given by (1) and S_2^q has Laplace transform

$$\phi_{S_2^q}(s) \sim \int_0^{\infty} e^{-sx^q} \cdot x^{2\beta-1} \cdot \left(\int_0^1 e^{-x\cdot(b_1\cdot(1-u)+b_2\cdot u)} \cdot (u\cdot(1-u))^{\beta-1}\, du \right) dx.$$

We will use Proposition 1 and recall the definition of the class HCM in Section 1. We compute $H_2 = \phi_{S_2^q}(st) \cdot \phi_{S_2^q}(\frac{s}{t})$ as a product of two such integrals and get

$$H_2 \sim \int_0^{\infty}\int_0^{\infty} e^{-sx^q\cdot(t\cdot y + t^{-1}\cdot y^{-1})} \cdot x^{2\beta-1}. \tag{5}$$

$$\int_0^1\int_0^1 e^{-x\cdot\left(b_1\cdot((1-u)\cdot y^{\alpha}+(1-v)\cdot y^{-\alpha})+b_2\cdot(u\cdot y^{\alpha}+v\cdot y^{-\alpha})\right)} \big(u(1-u)\cdot v(1-v)\big)^{\beta-1} dudv\, dx\frac{dy}{y},$$

after a hyperbolic change of variables $x \to x \bullet y$, $y \to x/y$ and a substitution $y \to y^{\alpha}$, $\alpha = 1/q$, and set out to prove that H_2 is CM wrt $t + t^{-1}$. We fix s and x and denote the inner integral by I_2. Then, after substitutions $u \to \frac{1}{1+u}$ and $v \to \frac{1}{1+v}$, $I_2 = \int\int e^{-E_1} \cdot g(u,v)dudv,$ where

$$E_1 = x \cdot \frac{b_1(u\cdot y^{\alpha} + v\cdot y^{-\alpha}) + b_2(v\cdot y^{\alpha} + u\cdot y^{-\alpha}) + (b_1uv + b_2)\cdot(y^{\alpha} + y^{-\alpha})}{(1+u)(1+v)}$$

and $g(u,v) = \frac{(uv)^{\beta-1}}{((1+u)(1+v))^{2\beta}}$ and the integration is over $(0\,;\infty) \times (0\,;\infty)$. We set out to prove that I_2 is CM wrt $y^{\alpha} + y^{-\alpha}$. Without loss of generality, we assume that $y > 1$ and put $y^{\alpha} + y^{-\alpha} = 2s$. Then we get

$$y^{\alpha} = s + \sqrt{s^2 - 1} \text{ and } y^{-\alpha} = s - \sqrt{s^2-1}$$

and note that also y^{α} is a Bernstein function wrt $y^{\alpha} + y^{-\alpha}$.

Next we define

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$$E_2 = b_1(u\cdot y^{\alpha} + v\cdot y^{-\alpha}) + b_2(v\cdot y^{\alpha} + u\cdot y^{-\alpha}) = (b_1u+b_2v)\cdot y^{\alpha} + (b_1v+b_2u)\cdot y^{-\alpha}$$

and denote

$$\Delta = (b_1u+b_2v) - (b_1v+b_2u) = (b_1-b_2)(u-v).$$

We observe that the integral I_2 is unchanged if b_1, b_2 and u,v are interchanged and $y \to y^{-1}$. The same is true for the integral I_2 if it is evaluated over any of the sets $\{\Delta > 0\}$ or $\{\Delta < 0\}$.

If $\Delta > 0$, we can rewrite E_2 as

$$E_2 = A\cdot(y^{\alpha}+y^{-\alpha}) + B\cdot y^{\alpha},$$

where $A > 0$ and $B \geq 0$ only depend on u,v,b_1 and b_2. It follows that E_2, and thereby also E_1, is a Bernstein function wrt $y^{\alpha} + y^{-\alpha}$ in this case.

In the opposite case $\Delta < 0$ we get $E_2 = A \bullet (y^{\alpha} + y^{-\alpha}) + B \bullet y^{-\alpha}$. It follows from the substitutions above that I_2 is unchanged wrt $y \to y^{-1}$ and we are back in the first case. We conclude that E_1 is a Bernstein function wrt $y^{\alpha} + y^{-\alpha}$ and I_2 is CM wrt $y^{\alpha} + y^{-\alpha}$.

Then I_2 is also CM wrt $y + y^{-1}$ by Bondesson [3], Ex. 4.3.4, p. 69, since $y^{\alpha} + y^{-\alpha}$ is a Bernstein function wrt $y + y^{-1}$. By Bernstein's Theorem, I_2 can be represented by a Laplace transform $I_2 = \int_0^\infty e^{-\lambda\cdot(y+y^{-1})}d\nu(\lambda)$, for a nonnegative Borel measure ν. Inserting this formula into H_2 then gives

$$H_2 \sim \int_0^\infty\int_0^\infty e^{-\left(sx^q\cdot(t\cdot y+t^{-1}\cdot y^{-1})+\lambda\cdot(y+y^{-1})\right)}\,\frac{dy}{y}\,d\nu(\lambda).$$

The exponent (with reversed sign) is a linear combination of y and y^{-1} and equals

$$y\cdot(sx^q\cdot t+\lambda)+y^{-1}\cdot(sx^q\cdot t^{-1}+\lambda) = \rho\cdot\left(s^2x^{2q}+\lambda^2+sx^q\lambda\cdot(t+t^{-1})\right),$$

after the substitution putting the second term equal to $1/\rho$. This proves that H_2 is CM wrt $t + t^{-1}$ and then $(X_1 + X_2)^q \sim GGC$ by Proposition 1, which completes the proof of Theorem 1 in the case $n = 2$.

Let $n \geq 3$ be an arbitrary integer and let S_n be the sum of the n independent gamma variables defined in (2). We start from the Laplace transform $\phi_{S_n^q}$ of S_n^q,

$$\phi_{S_n^q}(x) \sim \int_0^\infty e^{-sx^q}\cdot x^{n\beta-1}\cdot\int_0^1\cdots\int_0^1 e^{-x\cdot C_n(\mathbf{u})}\cdot B_n(\mathbf{u})\,du_1\cdots du_{n-1}\,dx.$$

In analogy with the case $n = 2$, we define $H_n = \phi_{S_n^q}(st)\cdot\phi_{S_n^q}\left(\frac{s}{t}\right)$ as a product of two such integrals and get in analogy with (5)

$$H_n \sim \int_0^\infty\int_0^\infty e^{-\left(sx^q\cdot(t\cdot y+t^{-1}\cdot y^{-1})\right)}\cdot x^{2n\beta-1}.$$

$$\int_0^1\cdots\int_0^1 e^{-x\cdot\left(C_n(\mathbf{u})y^{\alpha}+C_n(\mathbf{v})y^{-\alpha}\right)}\cdot B_n(\mathbf{u})\cdot B_n(\mathbf{v})\,du_1\cdots du_{n-1}dv_1\cdots dv_{n-1}\,dx\frac{dy}{y},$$

after a hyperbolic change of variables $x \to x \bullet y$, $y \to x/y$ and a substitution $y \to y^{\alpha}$. We denote the inner integral in H_n by I_n.

Now we assume that I_{n-1} is CM wrt $y^{\alpha} + y^{-\alpha}$ for any sum of $n - 1$ independent gamma variables. The last two integrals in I_n are equal to

$$J_n = \int_0^1\int_0^1 e^{-x\cdot E_n}\cdot\left((1-u_{n-1})\cdot(1-v_{n-1})\right)^{\beta-1}\left(u_{n-1}\cdot v_{n-1}\right)^{(n-1)\beta-1}\,du_{n-1}\,dv_{n-1},$$

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where by (3)

$$E_n = b_{n-1} \cdot \left((1-u_{n-1}) \cdot Ay^{\alpha} + (1-v_{n-1}) \cdot By^{-\alpha}\right) +$$
$$+ b_n \cdot \left(u_{n-1}Ay^{\alpha} + v_{n-1}By^{-\alpha}\right)$$

and

$$A = u_1 u_2 \cdots u_{n-2}, \;\; B = v_1 v_2 \cdots v_{n-2}.$$

We apply Lemma 2 with these values on A and B. Then J_n is CM wrt $Ay^{\alpha} + By^{-\alpha}$ and can be represented by a Laplace transform

$$J_n = \int_0^{\infty} e^{-\lambda \cdot \left(u_1 u_2 \cdots u_{n-2} \cdot y^{\alpha} + v_1 v_2 \cdots v_{n-2} \cdot y^{-\alpha}\right)} \, \nu(d\lambda),$$

for some nonnegative Borel measure ν. Now we insert J_n back into I_n. Then for every fixed $\lambda > 0$, In corresponds to I_{n-1} for a sum of $n - 1$ independent gamma variables and we conclude that I_n is CM wrt $y^{\alpha} + y^{-\alpha}$, by the induction hypothesis. Recalling that $y^{\alpha} + y^{-\alpha}$ is a Bernstein function wrt $y + y^{-1}$ and a substitution similar to the one used in the proof for the case $n = 2$ then proves that H_n is CM wrt $t + t^{-1}$. We conclude that $S_n^q \sim GGC$ and the proof of Theorem 1 is complete by Proposition 1.

Proof of Lemma 2. The proof follows the case $n = 2$ in the proof of Theorem 1, with $y^{\alpha}, y^{-\alpha}$ replaced by $Ay^{\alpha}, By^{-\alpha}$. For the readers convenience we sketch the proof. By (4) we must show that $L = \int_0^1 \int_0^1 e^{-\cdot E} \cdot g(u,v)\, du\, dv$, where

$$E = b_1\left((1-u)Ay^{\alpha} + (1-v)By^{-\alpha}\right) + b_2(uAy^{\alpha} + vBy^{-\alpha}),$$

is CM wrt $Ay^{\alpha} + By^{-\alpha}$. We start with substitutions $u \to \frac{1}{1+u}$ and $v \to \frac{1}{1+v}$ to get $L = \int\int e^{-E_1} g(u,v) du dv$, where

$$E_1 = x \cdot \frac{b_1(uAy^{\alpha} + vBy^{-\alpha}) + b_2(vAy^{\alpha} + uBy^{-\alpha}) + (b_1 uv + b_2)(Ay^{\alpha} + By^{-\alpha})}{(1+u)(1+v)}$$

and $g(u,v) = (uv)^{\beta_1 - 1} \cdot \left((1+u)(1+v)\right)^{-\beta_1-\beta_2}$ and the integration is over $(0,\infty) \times (0,\infty)$.

It is easy to see that L is unchanged if (b_1, b_2) and (u,v) are interchanged and $Ay^{\alpha} \to By^{-\alpha}$. Without loss of generality we assume that $Ay^{\alpha} > By^{-\alpha}$ and put $Ay^{\alpha} + By^{-\alpha} = 2s$. Then we get

$$Ay^{\alpha} = s + \sqrt{s^2 - AB} \;\text{ and }\; By^{-\alpha} = s - \sqrt{s^2 - AB}$$

and note that also Ay^{α} is a Bernstein function wrt $Ay^{\alpha} + By^{-\alpha}$.

Let E_2 denote the first two terms in the nominator of E_1, then we can rewrite E2 as

$$E_2 = (b_1 u + b_2 v) \cdot Ay^{\alpha} + (b_1 v + b_2 u) \cdot By^{-\alpha}$$

and $\Delta = (b_1 u + b_2 v) - (b_1 v + b_2 u) = (b_1 - b_2)(u - v)$. The rest of the proof is the same as in the case $n = 2$ and is left to the reader. We conclude that E_1 is a Bernstein function wrt $Ay^{\alpha} + By^{-\alpha}$ and L is CM wrt $Ay^{\alpha} + By^{-\alpha}$, which completes the proof of Lemma 2.

Remark 1. The old conjecture that $X \sim GGC$ implies $X^q \sim GGC$, $q > 1$, is a natural structural property of GGC mentioned in Bondesson [3], p. 97. A different approach was made in Bondesson [4], where the Laplace transform of S_n^q is expressed using the product of the densities of the individual rvs X_i, $1 \le i \le n$. The following sufficient condition for Theorem 1 to be true is given in Bondesson [4], Conjecture 2.

For every $q \ge 1$, $\alpha = 1/q$ and fixed positive numbers $u_1, u_2, \cdots, u_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$, the function

$$\int \frac{1}{v_1 v_2 \cdots v_n} e^{-E} d\mathbf{v}, \; where \; E = (\sum u_i v_i^{\alpha})^q + (\sum u_i v_i^{-\alpha})^q + \sum \lambda_i (\frac{t}{v_i} + \frac{v_i}{t})$$
$$is \;\; CM \;\; wrt \;\; t + t^{-1}.$$

Remark 2. The advantages with the method used here compared to the one in Remark 1 is that I_n is inductively defined, the exponent in the integrand of I_n is a linear function wrt y^{α} and $y^{-\alpha}$, the inner integral J_n has only two variables for all n and that

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the method of successive substitutions works here.

Remark 3. For $n = 2$ the integral I_2, with $c = 1$ can be expressed as a product of two Modified Bessel functions of first order using computer algebra to be

$$I_2 = \pi \cdot \Gamma(\beta)^2 \cdot e^{-(y+\frac{1}{y})} \cdot \mathrm{BesselI}(\beta, y/2) \cdot \mathrm{BesselI}(\beta, 1/2y).$$

A bold but natural suggestion is that Theorem 1 holds for more general compositions $f \circ X$, $X \sim GGC$, where f belongs to some class of smooth, increasing and convex functions defined on $[0,\infty)$ and satisfying $f(0) = 0$.

4. Applications

The class *GGC* is closed wrt sums and products of independent rvs and now also wrt q–th powers, $q > 1$. This gives the following result.

Theorem 2 Let $\{X_i\}_1^n$ *be independent rvs,* $X_i \sim GGC$*, and let* $q_i \geq 1$, $1 \leq i \leq n$. *Then* $\prod_1^n X_i^{q_i} \sim GGC \text{ and } \sum_1^n X_i^{q_i} \sim GGC.$

The class of extended generalized gamma convolutions (*EGGC*) was introduced by Thorin [12] and consists of limit distributions for sums of independent positive and negative gamma rvs. The symmetric distributions in *EGGC* are denoted by *symEGGC* and are characterized by $X \sim symEGGC$ if and only if $X = \sqrt{Y} \cdot Z$, for some $Y \sim GGC$ and independent $Z \sim N(0,1)$, see Bondesson [4], Ch. 5 or Steutel, van Harn [9], Ch. VI, §11. Theorem 1 implies that, if $0 < \alpha < 1$, every $Y \sim GGC$ can be written $Y = Z^{\alpha}$, for some $Z \sim GGC$. This gives the following extension of Bondesson [4], Theorem 2.

Theorem 3 *Let* $0 < \alpha \leq 2 \leq \beta$*, then*
(a) *If* $Y \sim GGC$ *and* $Z \sim N(0,1)$ *are independent, then* $Y^{1/\alpha} \bullet Z \sim symEGGC$.
(b) *If* $X \sim symEGGC$ *there exist* $Y \sim GGC$ *and an independent* $Z \sim N(0,1)$ *such that* $X = Y^{1/\beta} \bullet Z$.

If $\mathcal{A}$ and $\mathcal{B}$ are two classes of distributions, we let $\mathcal{A} \times \mathcal{B}$ denote the class of products $X \bullet Y$ of independent rvs $X \sim \mathcal{A}$ and $Y \sim \mathcal{B}$. Then we can express

Theorem 3 as

$$GGC^{1/\alpha} \times N(0,1) \subseteq symEGGC \subseteq GGC^{1/\beta} \times N(0,1),$$

$0 < \alpha \leq 2 \leq \beta$, with equality for $\alpha = \beta = 2$.

We finally give a new proof of Bondesson [4] Theorem 3, see the comment on p. 1075.

Theorem 4 *If* $X \sim GGC$ *has left extremity* $a \geq 0$*, then* $e^X - e^a \sim GGC$.

Proof. If $a = 0$, $X \sim GGC$ and $0 < r < 1$, then $(1+rX)^{\frac{1}{r}} \sim GGC$ by Theorem 1 and

$$Pr((1+rX)^{\frac{1}{r}} \leq 1+u) = Pr(X \leq [(1+u)^r - 1]/r) \to Pr(e^X \leq 1+u),$$

as $r \to 0$, by L'Hopital's rule. Hence $e^X - 1 \sim GGC$, since *GGC* is closed wrt weak limits. If $a > 0$ we have $e^X - e^a = e^a(e^{X-a} - 1)$, $(X - a) \sim GGC$ has left extremity zero and $e^X - e^a \sim GGC$ follows from the first case [7].

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